MAT511 homework, Sept. 10, 2003

(1) As we did in class, consider the following axiomatic system describing a system of people and clubs. We are considering a finite, nonempty collection S of people, and want to describe the clubs to which they belong.

Definition: A club is a non-empty set of people which satisfies certain axioms as outlined below.

Definition: If two clubs have no members in common, they are called conjugate clubs. If $\mathcal{C}$ is our club, we can use $\overline{\mathcal{C}}$ to denote a club which is conjugate to $\mathcal{C}$.

Axiom 1: Every person of S is a member of at least one club. (If p is a person in club $\mathcal{C}$, we can use the notation $p \in \mathcal{C}$ to express this fact.)

Axiom 2: For every pair of people of S, there is one and only one club to which they both belong.

Axiom 3: For every club, there is one and only one conjugate club. (Note that as a consequence of this axiom, the notation $\overline{\mathcal{C}}$ is not ambiguous. Without this axiom, $\overline{\mathcal{C}}$ might refer to two different clubs, or no clubs at all.)

Using these axioms and definitions, we proved the following

Theorem 1. *Every person of S is a member of at least two clubs.*

Here is how the proof we came up with went, pretty much:

Proof. Let p be an arbitrary person in S. By Axiom 1, there is a $\mathcal{C}$ with $p \in \mathcal{C}$.

By Axiom 3, there is another club $\overline{\mathcal{C}}$ which is conjugate to $\mathcal{C}$. By the definition of conjugate, p is not a member of $\overline{\mathcal{C}}$, and since clubs are nonempty, there must be some other person q for which $q \in \overline{\mathcal{C}}$, and also $q \notin \mathcal{C}$.

Now, by Axiom 2, both p and q must be in *some* club together, but this is neither $\mathcal{C}$ (since $q \notin \mathcal{C}$) nor is it $\overline{\mathcal{C}}$ (since $p \notin \overline{\mathcal{C}}$). Thus, there is a third club $\mathcal{D}$, with $p \in \mathcal{D}$. So p is in at least two different clubs, namely $\mathcal{C}$ and $\mathcal{D}$.

Since p was chosen arbitrarily in S, we have shown that each person in S is a member of at least two clubs. □

You should prove the following three theorems in this system. You can, of course, use things you have already proven in your proofs.

Theorem 2. *Every club contains at least two members.*

Theorem 3. *S contains at least four people.*

Theorem 4. *There exist at least six different clubs.*

(2) During the first class, we proved that $\sqrt{2}$ is irrational. Write a proof that $\sqrt{3}$ is irrational, using a similar argument.

In case you forgot, what we did was to assume that there was a rational number p/q in lowest terms for which

$$\left(\frac{p}{q}\right)^2 = 2$$

and arrived at the conclusion that both p and q are even, contradicting the assumption that p/q were in lowest terms.

(3) Section 1.5 in the 4th edition of the text (I passed out xeroxes), problems **2**, **5**, and **8abc**. If you have the 5th edition, this is section 1.6, problems problems **2**, **5**, and **7abc**.

(4) §1.6 in 4th edition of the text, problems **1**, **3**. In the 5th edition, this is section 1.7, problems **1**, **3**.

I'd like you to try all the problems. If you run short of time, I am most interested in your work on the first two problems. Let me know on your homework sheet how much time this assignment took you, so I can gauge things.