Second Exam MAT/CSE 371, Spring 2000

Directions: This is a take-home exam. This means you can use whatever books and materials you like, excluding your classmates. I expect you to do your own work. If you have trouble understanding something, ask me either in person or via email.

Please write up each question neatly, and clearly mark which question or part is being answered. Do not try to save on paper by cramming everthing together– a well-organized, readable exam is likely to be looked upon more fondly than a cramped messy one when looking to assign partial credit.

1. Let $\mathbb{G}$ be a theory with equality having the usual logical axioms and four predicates: the unary predicates P and L, and the binary predicates $=$ and $\ominus$. Let $\mathbb{G}$ have the following non-logical axioms:

G1. $P(x) \Rightarrow \neg L(x)$

G2. $x \ominus y \Rightarrow P(x) \wedge L(y)$

G3. $L(x) \Rightarrow (\exists y)(\exists z)(y \neq z \wedge y \ominus x \wedge z \ominus x)$

G4. $P(x) \wedge P(y) \wedge x \neq y \Rightarrow (\exists_1 z)(L(z) \wedge x \ominus z \wedge y \ominus z)$

G5. $(\exists x)(\exists y)(\exists z)(P(x) \wedge P(y) \wedge P(z) \wedge \neg\,[(\exists u)(L(u) \wedge x \ominus u \wedge y \ominus u \wedge z \ominus u)]$

Interpret $\mathbb{G}$ as a (sub)theory of planar geometry, and interpret the predicates as follows:

- $P(x)$ means x is a point.
- $L(x)$ means x is a line.
- $x = y$ means x and y are identical.
- $x \ominus y$ means x is a point lying on y.
- Also, let $u \,\|\, v$ stand for the predicate $L(u) \wedge L(v) \wedge \neg(\exists w)(w \ominus u \wedge w \ominus v)$, which is interpreted as "*u and v are distinct parallel lines.*" This is needed for parts **d** and **e** only.

Using this interpretation, do each of the following. Note that each part can be done independently of the others.

a. Translate each of the non-logical axioms into ordinary geometric language.

b. Translate the following theorem into ordinary geometric language:

$$\vdash (\forall u)(\forall v)\ [L(u) \wedge L(v) \wedge u \neq v \Rightarrow (\forall x)(\forall y)(x \ominus u \wedge x \ominus v \wedge y \ominus u \wedge y \ominus v \Rightarrow x = y)]$$

Then provide a formal deduction of the theorem from the axioms. (Note that this proof must not explicitly use "ordinary geometric reasoning", although of course you may use it to figure out how to construct the proof).

c. Give a formal proof that $\vdash u \,\|\, v \Rightarrow u \neq v$.

d. Translate the following sentence into ordinary language, and show that there exists a normal model of $\mathbb{G}$ with a finite domain in which it is true.

$$(\forall x)(\forall y)\,[P(x) \wedge L(y) \wedge \neg(x \ominus y) \Rightarrow (\exists_1 z)(x \ominus z \wedge z \,\|\, y)]$$

Note that this only requires you to give a domain which is a model for $\mathbb{G}$, not prove the statement. Be sure to explicitly enumerate the domain, and explicitly show that it is a model.

e. Show that there exists a model of $\mathbb{G}$ in which the following is true:

$$(\forall x)(\forall y)\,(L(x) \wedge L(y) \wedge x \neq y \Rightarrow \neg(x \,\|\, y))$$

2. *Compactness of models.* Let K be a theory such that all finite subsets of the set of axioms have models. Show that K also has a model.

3. Show that the following wf is true for all finite domains, but there is an infinite domain in which it is false:

$$(\forall x)(\forall y)(\forall z)\Big[A(x,x) \wedge \Big(A(x,y) \wedge A(y,z) \Rightarrow A(x,z)\Big) \wedge \Big(A(x,y) \vee A(y,x)\Big)\Big] \Rightarrow (\exists y)(\forall x)A(y,x)$$

4. Show that the "method of infinite descent" can be deduced for the axioms of the theory S (our axiomatization of arithmetic). That is, prove that

$$\vdash_S (\forall x)\Big(\mathcal{B}(x) \Rightarrow (\exists y)(y < x \wedge \mathcal{B}(y))\Big) \Rightarrow (\forall x)\neg\mathcal{B}(x)$$

5. Express the following wfs in $\mathcal{L}_A$, the language of arithmetic used in chapter three. If you use abbreviations such as $\overline{2}$ for $0''$ (please do, to improve readability), spell them out explicitly.

a. $y = 2^x$

b. 9 is the largest even prime number.

c. Goldbach's Conjecture: Every even number greater than 2 can be written as the sum of two odd primes. (Extra bonus question: prove Goldbach's Conjecture. If you do this one, not only will I give you an A in the course, but there is a million dollar prize you can claim.)

d. z is a perfect number. (A number is called "perfect" if it is equal to the sum of its divisors, excluding itself. For example, 6 is perfect because $6 = 1 + 2 + 3$.)