1. Let $X$ and $Y$ be topological spaces, and let $f : X \to Y$ be continuous and surjective. Show that if $X$ is connected, then $Y$ must be connected.

If $Y$ is not connected there exist two open, non-empty sets $A$ and $B$ so that

$A \cup B = Y$  \hspace{1cm} A \cap B = \emptyset$

- Since $f$ is continuous $f^{-1}(A)$ and $f^{-1}(B)$ are open
- Also $f^{-1}(A) \cap f^{-1}(B) = \emptyset$ since $f$ is a function
  (i.e. if $x \in f^{-1}(A) \cap f^{-1}(B)$ then $f(x) \in A$, $f(x) \in B$, but $A \cap B = \emptyset$)
- $f^{-1}(A) \cup f^{-1}(B) = X$ since $f$ is surjective
  (i.e. every $y \in Y$ has some $x \in X$ so that $f(x) = y$ such a $y$ is in $A$ or $B$ since $A \cup B = Y$)
- Also $f^{-1}(A) \neq \emptyset$ since $A \neq \emptyset$ and $f^{-1}(B) \neq \emptyset$ since $B \neq \emptyset$.

Therefore $X$ is not connected.
2. Let $M$ be the image of $\mathbb{R}^2$ under $h : \mathbb{R}^2 \to \mathbb{R}^3$ given by $h(x, y) = (x^3, x^2, y)$

   a) Calculate $Dh$.
   
   $$Dh = \begin{pmatrix} 3x^2 & 0 \\ 2x & 0 \\ 0 & 1 \end{pmatrix}$$

   b) What is the tangent space to $M$ at $(0, 0, 0)$?
   
   $$Dh_{(0,0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{R}$$

   c) What is the tangent space to $M$ at $(1, 1, 1)$?
   
   $$Dh_{(1,1,1)} = \begin{pmatrix} 3 & 0 \\ 2 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{R}^2$$

   d) Is $M$ a smooth manifold? Justify your answer.
   
   No, $M$ is not a smooth manifold because the derivative matrix is not of full rank at $(0,0,0)$. 


3. The set of all $2 \times 2$ matrices with real entries and determinant 1 is called $SL_2(R)$. Show that $SL_2(R)$ is a smooth 3-manifold by giving charts for it. Be sure to justify that your charts cover $SL_2(R)$ and that they are smooth.

[Note: We can show that $SL_2(R)$ is a manifold since det is a smooth function and $SL_2(R)$ is the preimage under det.]

$$M_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in R \right\}$$ is diffeomorphic to $R^4$.

$$\text{det} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \text{ then } ad - bc = 1$$

Solve for $b$ then $b = \frac{ad - 1}{c}$ if $c \neq 0$

The chart for $c \neq 0$ from $R^3 \to SL_2(R)$ is

$$(a, c, d) \to \begin{pmatrix} a & \frac{ad - 1}{c} \\ c & d \end{pmatrix}$$

This chart is continuous and smooth.

The chart for $b \neq 0$

$$(a, b, d) \to \begin{pmatrix} a & b \\ \frac{ad - 1}{b} & d \end{pmatrix}$$

This chart is continuous and smooth.

If $b = 0$ and $c = 0$

$$(a, 0, d) \to \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \text{ with } d = \frac{1}{a} \text{ and } a \neq 0.$$ This chart is continuous and smooth.
4. Consider the two disjoint, closed disks

\[ D_1 = \{(x, y) \mid x^2 + y^2 \leq 1\} \text{ and } D_2 = \{(x, y) \mid (x - 4)^2 + y^2 \leq 1\} \]

Let \( M = D_1 \cup D_2 \).

Suppose \( f : M \to M \) a smooth function. Show that \( f \) must have a period 2 point; that is, there must be a point \( p \in M \) such that \( f(f(p)) = p \).

Case I:

a) \( f(D_1) \subset D_1 \)

For this case we ignore \( D_2 \).

Apply Brouwer’s Fixed Point Theorem, then there exists \( p \) such that \( f(p) = p \) then \( f(f(p)) = f(p) = p \).

b) \( f(D_2) \subset D_2 \), then \( p \in D_2 \).
Case II: If neither Case I a) nor I b) hold, since $f$ is continuous the following must hold

- $f(D_1) \subset D_2$ then $f(f(D_1)) \subset f(D_2) \subset D_1$ so $f^2 : D_1 \to D_1$
  There exists $p \in D_1$ with $f(f(p)) = p$

- $f(D_2) \subset D_1$ then $f(f(D_2)) \subset f(D_1) \subset D_2$ so $f^2 : D_2 \to D_2$
  There exists $p \in D_2$ with $f(f(p)) = p$