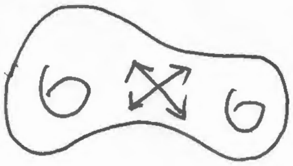
Dec. 2. by Keren Wang

$m^n \subset \mathbb{R}^k$, compact manifold, $\partial m = \varnothing$

$\dim m = n < k$



vector field V on m w/ isolated zeros z_i

From before, if N ($\dim N = k$) is manifold w/,

$\partial N \neq \phi$, $\partial N = \cup S^{k-1}$

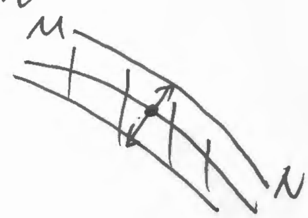
vector field w on N with iso zeros and pointing outward on each boundary component.

Then $\sum_{z \in N} \lambda = \deg(\text{Gauss map})$

Start w/ M, fatten m to N. i.e. Let ε small enough

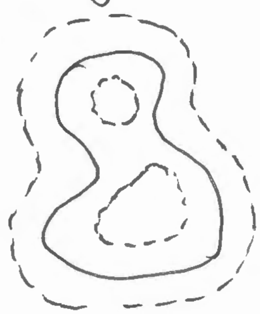
and $N_\varepsilon = \{ y \in \mathbb{R}^k \mid \|x - y\| < \varepsilon, \text{ for } x \in M \}$

(HW. Prove that N is a k-manifold w/ ∂).



we must choose ε small enough.

If ε is too big, it might be



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we want a vector field $\vec{w}$ on N_ε, that tells us something about $\vec{v}$ on M.

We need $\vec{w}$ isolated zeros; $\vec{w}$ outward pointing on ∂N_ε

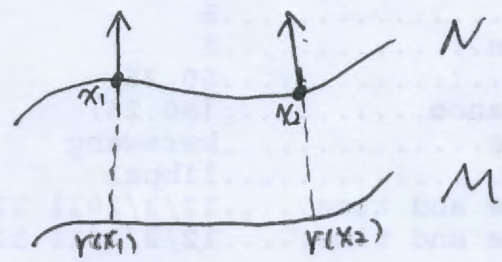
Just like

(Sorry, it's not a good picture.)

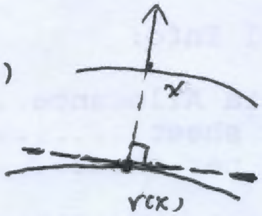
On N_ε, we define.

for each $x \in N$, $r(x)$ to be the ~~cot~~ closest point of M.

(i.e. $r(x)$ is a retraction of N_ε onto M)



Notice that, $\overrightarrow{x-r(x)}$ is orthogonal to $TM_{r(x)}$



$\varphi(x) = \|x - r(x)\|^2$

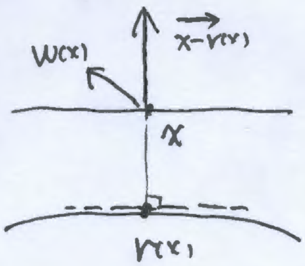
$\nabla\varphi(x) = 2(x-r)$ (as $\|x-r(x)\|$ is a const).

So for any $x \in \partial N_\varepsilon$, $\nabla\varphi(x)$ is outward normal to ∂N_ε

Define $g(x) = \frac{\nabla\varphi}{\|\nabla\varphi\|} = \frac{x-r}{\varepsilon}$

$\vec{w}(x) = ((x - r(x)) + \vec{V}(r(x)))$ (Here we extend v to vector field w on N

Since $g \cdot W = \varepsilon$, $\vec{w}$ points outward on ∂N_ε



Only zeros of $\vec{w}$ are (exactly) zeros of $\vec{v}$. so $\vec{w}$ is a vector field satisfying conditions of Lemma.

What is dW at a zero Z of W?

$$dw = \left(\begin{array}{c|c} dV & 0 \\ \hline 0 & \underbrace{1}_{=\frac{d}{dx}(x-r)} \end{array}\right),$$

thus $\det(d(w(z))) = \det(d(v(z)))$

so $\Sigma\iota$ is same for $\vec{v}$ as $\vec{w}$ #.

$\left[\ \iota(z) = \mathrm{sign}(\det(dW(z))) = \mathrm{sign}(\det(dV(z)))\ \right]$