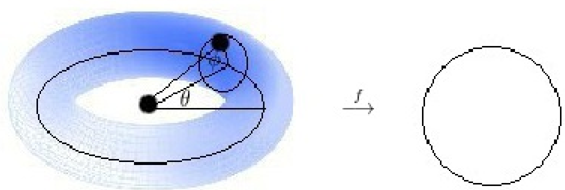
Notes for MAT 364, 10/05/2011

Midterm on October 31[st].

Lemma: $f : M \to N$, dim $M = m$, dim $N = n$, $m \geq n$, and f is smooth. If $y \in N$ is a regular point of N, then $f^{-1}(y)$ is a $(m-n)$ dimensional sub-manifold.

Example: Consider the torus $T^2 = \{(\theta, \phi) | 0 \leq \theta, \phi \leq 2\pi\}$ and S^1 and the map $f : T^2 \to S^1$ where $f(\theta, \phi) = \theta$.



Then, $f^{-1}(\theta_0) = \{(\theta_0, \phi), 0 \leq \phi \leq 2\pi\}$.

Let $x \in f^{-1}(y)$ (that means $f(x) = y$), and y is regular. The map $df_x : TM_x \to TN_y$ is onto. TM_x is a m dimensional vector space and TN_y is a n dimensional vector space.

Then, $\ker(df_x)$ is a vector space of dimension $(m-n)$.

In our example, our kernel for df_x is 1 dimension.

Note that $M \subseteq \mathbb{R}^k$ $(k \geq m)$. Make $F : \mathbb{R}^m \to \mathbb{R}^m$ such that $F(x_1, \ldots, x_m) = (f(x_1, \ldots, x_m), L(x_1, \ldots, x_m))$.

Since $f : \mathbb{R}^m \to \mathbb{R}^n$, $f(x_1, \ldots, x_m)$ has n components and we can pick linear map $L : \mathbb{R}^m \to R^{m-n}$. Then, $(f(x_1, \ldots, x_m), L(x_1, \ldots, x_m))$ has m components. Thus, now we have extended the map $f : \mathbb{R}^m \to \mathbb{R}^n$ to the map $F : \mathbb{R}^m \to \mathbb{R}^m$.

Now, $dF = (df_x, dL)$ where df_x has n components and dL has $(m-n)$ components. As a matrix, df_x will have the form:

$$\mathbf{n}\left\{ \overbrace{\begin{pmatrix} df \\ \hline dL \end{pmatrix}}^{\mathbf{m}} \right\} \mathbf{m-n}$$

We can now change dF to Jordan From by changing choosing a different basis, so that

$$df_x = \left(\begin{array}{ccc|c} a_{1,1} & \cdots & a_{1,n} & \\ & \ddots & \vdots & 0 \\ 0 & & a_{n,n} & \end{array} \right)$$

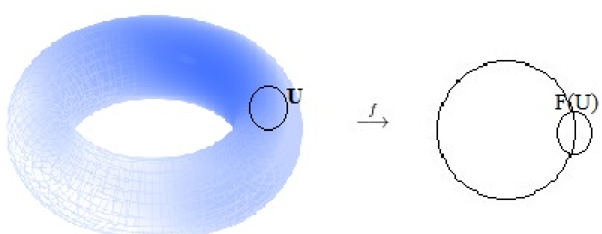
and we can select dL such that under this basis,

$$dL = \left(\begin{array}{c|ccc} & b_{1,1} & \cdots & b_{1,(m-n)} \\ 0 & & \ddots & \vdots \\ & & & b_{(m-n),(m-n)} \end{array}\right)$$

$$\text{Then, } dF = \left(\begin{array}{ccc|ccc} a_{1,1} & \cdots & a_{1,n} & & & \\ & \ddots & \vdots & & 0 & \\ 0 & & a_{n,n} & & & \\ \hline & & & b_{1,1} & \cdots & b_{1,(m-n)} \\ & 0 & & & \ddots & \vdots \\ & & & & & b_{(m-n),(m-n)} \end{array}\right).$$

Now $\ker(F) = 0$. Hence, we can invert F. Also, dF is an isomorphism from TM_x to $\mathbb{R}^m$ and is non-singular.

Hence, $F^{-1}(y)$ is a function.



$F : f^{-1}(y) \cap U \to y \times \mathbb{R}^{n-m}$ with F being onto.

The name of the other $(m-n)$ dimensions is normal vectors (or cotangent space).

Easy example:

$S^{n-1} = \{(x_1, \ldots, x_n) | {x_1}^2 + \ldots {x_n}^2 = 1\}$

The easy way to see that S^{n-1} is a smooth manifold is consider:

$f : \mathbb{R}^n \to \mathbb{R}$, $f(x_1, \ldots, x_n) = {x_1}^2 + \ldots {x_n}^2$.

Then, $f^{-1}(1) = S^{n-1}$.

Any $y \in \mathbb{R}$ where $y \neq 0$ is a regular value. We can apply lemma to show that S^{n-1} is smooth manifold.

Aside: Suppose TM_x is $\mathbb{R}^l$ and M is a smooth manifold, then dim $M = l$.

Suppose g_x is a chart around x, then dg_x is an isomorphism of vector space. Hence, $dg_x : \mathbb{R}^n \to \mathbb{R}^n$, so $n = l$.