Topology notes

October 4, 2011

1 Fundamental Theorem of Algebra

(I). Consider the stereographic projection mapping $h_+ : \mathbb{S}^2 \setminus \{(0,0,1)\} \longrightarrow \mathbb{C}$. This map h_+ takes points in the northern hemisphere of the sphere to points in the complex plane. The point $(0,0,1)$ is the point at infinity; as we get closer and closer to the this point h_+ maps points in $\mathbb{S}^2$ to points in the plane near infinity. Furthermore, it is clear that the map is bijective.

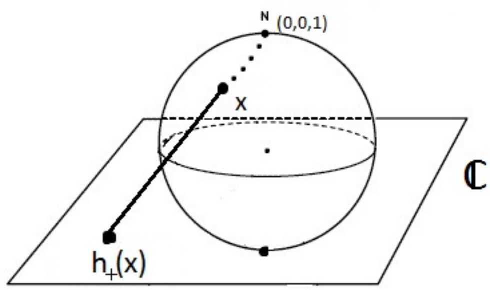

Figure 1: Stereographic Projection

(II). We define analogously the map h_- that maps points from the southern hemisphere onto the inside of the sphere. Notice that the point in the south pole is mapped onto the origin of the plane; $(0,0,-1) \longrightarrow (0,0) \in \mathbb{C}$ by h_-.

(III). We now need to find a map p that we can think of it as acting on the sphere. consider the following diagram:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{P} & \mathbb{C} \\ \uparrow h_+ & & \uparrow h_+ \\ \mathbb{S}^2 & \xrightarrow{p} & \mathbb{S}^2 \end{array}$$

we see that P in the diagram corresponds to the map p, where $p : x \longrightarrow h_+^{-1} P h_+(x)$ excluding the north pole (0,0,1), but let $p(0,0,1) = (0,0,1)$. This map p is smooth even in a neighborhoud of (0,0,1). To see this, we set $Q(z) = h_- p h_-^{-1}(z)$. Let now $f = h_+ h_-^{-1} : \mathbb{C} \longrightarrow \overline{\mathbb{C}}$. Now, notice that $f(re^{i\varphi}) = (1/r)e^{i\varphi}$. therefore, f maps $z \longrightarrow 1/\overline{z}$

(IV). Now, if $P(z) = a_0 z^n + a_1 z^{n-1} + ... + a_n$ ($a_0 \neq 0$), then we obtain that $Q(z) = z^n / (\overline{a_0} + \overline{a_1} z + ... + \overline{a_n} z^n)$. Therefore, Q is smooth in a neighborhoud of 0, and hence we have that $p = h_-^{-1} Q h_-$ is smooth in a neighborhoud of (0,0,1).

(V). Next notice that p has only a finite number of critical poins; for P fails to be a local diffeomorphism only at zeros of the derivative $P' = a_o n z^{n-1} + ... + a_{n-1}$, and there are only finitely many zeros since P' is not identically zero. The set of regular values of p, being a sphere with finitely many points removed, is therefore connected. Hence the locally function $\sharp p^{-1}(y)$ must be constant in this set. Since $\sharp p^{-1}(y)$ cant be zero everywhere, we conclude that it is zero nowhere. Thus p is surjective, and $P(z)$ must have a zero, and we conclude the proof of the fundamental theorem of algebra.

Note that this argument actually proves more: not only is there one zero, there are at most n zeros, where n is the degree of p. We can see this as follows.

Look at the map $Q(w) = w^n / (a_0 + a_1 w + \ldots + a_n w^n)$. For w close to zero, $Q(w) = w^n + \epsilon$, where ϵ is a small complex number. For c some small complex number which is a regular value of Q, $Q(w) = c$ will have n distinct solutions, each approximately equal to an n-th root of c. This tells us that $\sharp Q^{-1}(c) = n$. But this also tells us for y large, $\sharp p^{-1}(y) = n$.

Putting this together with the previous gives us the stronger form of the fundamental theorem of algebra: every complex polynomial of degree n has at most n zeros.