Topology Notes

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September 26, 2011

We're still trying to prove that Fundamental Theorem of Algebra, and we need a little bit more definitions.

Fundamental Theorem of Algebra 1. *Every polynomial $p : \mathbb{C} \to \mathbb{C}$ has a zero $z \in \mathbb{C}, p(z) = 0$.*

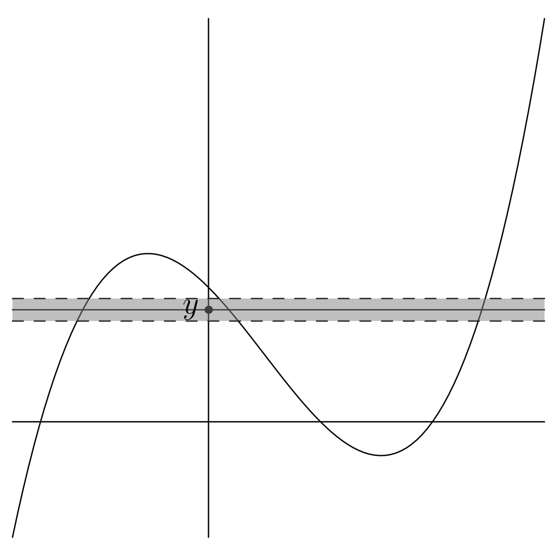
Last time, we were talking about Regular Values and Regular Points.

Definition 1. *For any function f, a point x is a **Regular Point** if the derivative df_x is nonsingular. A point y is a **Regular Value** if all the points in the pre-image $f^{-1}(y)$ are regular points.*

*Let us also define this operator: $N_y = \#f^{-1}(y)$, that is, the cardinality of that pre-image. This is sometimes called the **local topological degree**.*

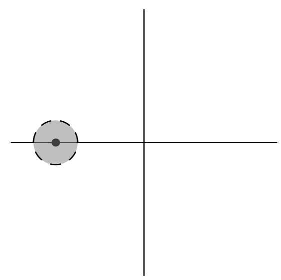
Proposition 1. *If $f : U \to \mathbb{R}^m$ is smooth and $y \in \mathbb{R}^m$ is a regular value for f, then $y \mapsto N_y$ is constant on a neighborhood of y. (That is, it's **locally constant**).*

For the below graph, the local topological degree is 3 at $y = 2$

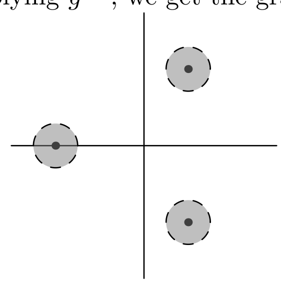


Notice that, if we move the line a little bit up or down, it still hits 3 points. That's what we mean when we say "locally constant."

So, for an example of the above proposition, let's look at $g : z \mapsto z^3$. For all $w \in \mathbb{C}, w \neq 0, N_w = 3$, but $N_0 = 1$. Look at the below example, where $w = -1$.



After applying g^{-1}, we get the graph below



So how do we prove this proposition? Let's use the Inverse Function Theorem (IFT)!

For $f : \mathbb{R} \to \mathbb{R}$, the IFT says: if y is a regular value, then there is a neighborhood W of x so that f has an inverse on that neighborhood, or $f : W \to \mathbb{R}$ is invertible.

Let's have $f(x_1) = y$. By IFT, there's a neighborhood $V_1(y)$ so that $f_1^{-1} : V_1 \to \mathbb{R}$ is defined and $f_1^{-1}(y) = x_1$. Let's have a set Bob $= \{x_i : f(x_i) = y\}$, so $N_y =$ Bob. We have $V_1, V_2, V_3, \ldots$ and for each $x_i \in$ Bob. Let $V = \cap V_i$. So, we have $f_1^{-1}, f_2^{-1}, f_3^{-1}, \ldots$ all defined on V. That means that N_y is locally constant. Check out page 8 of the Milnor for more on the exciting world of Regular Points and Regular Values.

We need one more concept in order to prove the FTA. We need to understand connectedness.

Definition 2. *A set S is* ***connected*** *if, whenever S is divided into two non-empty dis joint sets so that*

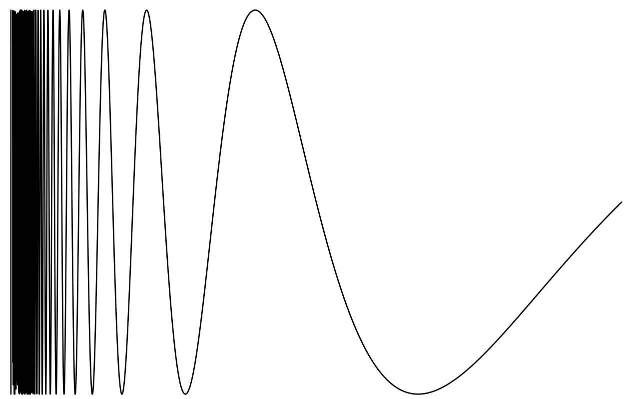
$$S = A \cup B, A \neq \emptyset, B \neq \emptyset, A \cap B = \emptyset,$$

either A or B contains a limit point of the other. (Equivalently, it's impossible to find two non-empty open sets $A, B \subseteq S$ so that $A \cap B = \emptyset, A \cup B = S$.)

A ***path*** *is the image of a continuous function $\gamma : [0,1] \to A$. (We will refer to γ as either a function or the image of that funtion. I will attempt to do so* *unambiguously.) A set A is* ***path-connected*** *if, for each $x, y \in A$, there exists a path γ such that $\gamma(0) = x, \gamma(1) = y$, and Image$(\gamma) \subseteq A$. As a little proposition, if something's path connected, it is also connected (This is homework problem 1a; I give you two definitions of connected to help out with the proof.)*

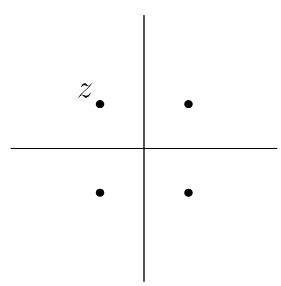
For our purposes, we'll use the ideas of path and path connected, but not connected. The above definition of "connected" can be found in the Kinsey book "Topology of Surfaces," although its equivalent reformulation was discussed in class. I recommend mulling it over, drawing a couple pictures to see how that kind of connectedness works.

Speaking of pictures, consider this pretty picture. Let $G_1 = \{(x, \sin(\frac{1}{x}))\}$ and let $G_2 = \{(0, y) : -1 \leq y \leq 1\}$, and let $G = G_1 \cup G_2$.

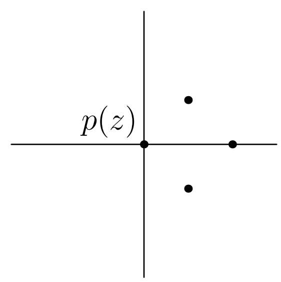


We can see that, clearly, G_1 and G_2 are path connected. However, G is not, because we can't bridge that infinitely small gap between $\sin \frac{1}{x}$ and the y-axis. That being said, this graph is connected. Recall the definition of connected says that the two sets make up G (and they do by definition), they're both non-empty, and their intersection is empty, but the limit points of either set are in one or the other. This is correct, as all the interior points are in G_1, and its frontier points at the y-axis are in G_2. The second definition doesn't really help much in this situation, but it might in others.

Let's get back to polynomials. The polynomial $p : \mathbb{C} \to \mathbb{C}$ has degree d. We want to prove that it has at most $d - 1$ critical points. Let z be a zero of p. We draw in the complex plane:



After applying the polynomial to those points, we get:



Here's the proof that it has at most $d-1$ critical points. Either 0 is a regular value or not. If it's not, then there's a critical point c so that $p(c) = 0$. Done, that's a zero. Otherwise, if 0 is a regular value, since there are $d-1$ critical values, we can find $x_0, y_0 \in \mathbb{C}$ so that $y_0 \neq 0, f(x_0) = y_0$. Let's take out those critical points from the complex plane; it'll still be path connected. So, connect y_0 to 0 by a path $\gamma(t)$, avoiding those critical value chasms. $\#p^{-1}$ is locally constant along this path $\gamma(t)$. (This is called *Analytic Continuation*).

Notice that $p : \mathbb{C} \to \mathbb{C}$ is onto. Well, that's about it for this set of notes. However, one last theorem: Sard's Theorem:

Sard's Theorem 1. *If we have $f : M \to N$, with f smooth, $C = \{$ critical points of $f\}$, then $f(C)$ has measure 0. That is to say, for all $\varepsilon > 0$, there exist countably many disks $D_\varepsilon \subseteq N$ that covers $f(C)$, their union having an area $< \delta$ for any $\delta > 0$.*

This means that, if we look at the image of our critical points, they'll end up like a dot on a line or a line in a plane or a plane in space. They won't have that "length" or "area" or "volume" concept defined on the space they are contained within.