MAT364, Homework 5
due wednesday 10/19

1. In the proof of the Fundamental Theorem of Algebra, we showed that if $f : \mathbb{C} \to \mathbb{C}$ is a polynomial, then the number of preimages of every regular point is the same (and is equal to the degree of f).

   If $G : \mathbb{R}^2 \to \mathbb{R}^2$ is a polynomial map (that is, $G(x, y) = (g_1(x, y), g_2(x, y))$, where g_1 and g_2 are polynomials in x and y), the result need not be true, even if G is surjective. Give an example of a surjective polynomial from $\mathbb{R}^2$ to $\mathbb{R}^2$ for which the result fails, and explain what step of the proof does not apply.

2. In the proof of the (smooth) Brouer Fixed Point Theorem, we had the following situation:

   The map g was a smooth diffeomorphism from $\bar{\mathbb{D}}$ to $\bar{\mathbb{D}}$ (where $\bar{\mathbb{D}}$ is the closed n-dimensional ball in $\mathbb{R}^n$, with boundary S^{n-1}) so that $g(x) \neq x$. We then defined $f : \bar{\mathbb{D}} \to S^{n-1}$ to be the map so that $f(x)$ lies on intersection of the line connecting $g(x)$ to x and S^{n-1}, and so that either $x = f(x)$ or x lies between $f(x)$ and $g(x)$ on this line.

   Write an explicit formula for $f(x)$, and show that it is smooth by calculating df_x.

3. Recall the construction of the middle-thirds Cantor set $\mathcal{C}$.

   - Let $\mathcal{C}_0$ be the interval $[0, 1]$.
   - Let $\mathcal{C}_1$ be the union of the intervals $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, that is $\mathcal{C}_0$ with the middle third removed.
   - In general, obtain $\mathcal{C}_n$ from $\mathcal{C}_{n-1}$ by removing the middle third of each interval in $\mathcal{C}_{n-1}$. Specifically

     $$\mathcal{C}_n = \frac{1}{3}\mathcal{C}_{n-1} \cup \left(\frac{2}{3} + \frac{1}{3}\mathcal{C}_{n-1}\right)$$

     (here, if X is a set of real numbers, $a + bX$ means $\{a + bx \mid x \in X\}$).
   - Then $\mathcal{C} = \displaystyle\bigcap_{n=1}^{\infty} \mathcal{C}_n$.

   As noted in class, $\mathcal{C} = \left\{ \displaystyle\sum_{k=1}^{\infty} \frac{a_k}{3^k} \;\middle|\; a_k \in \{0, 2\} \right\}$.

   (a) Show that the middle-thirds Cantor set has measure 0 by showing that the length of the intervals removed is 1. Specifically, let $M_n = \mathcal{C}_{n-1} \setminus \mathcal{C}_n$ be the intervals removed at the n^{th} stage, and $|M_n|$ be its total length. Observe that $M_i \cap M_j = \emptyset$ if $i \neq j$, and show that $\sum_{n=1}^{\infty} |M_n| = 1$.

   (b) Given any number α with $0 < \alpha < 1$, modify the construction of $\mathcal{C}$ to create a similar set $\mathcal{C}_\alpha$ which has measure α (More specifically, that its complement has total length $1 - \alpha$.) (Hint: adjust the size of the gaps at each stage).

   (c) Show that for each set $\mathcal{C}_\alpha$ constructed in (3b), there is a homeomorphism $f : [0, 1] \to [0, 1]$ so that $f(\mathcal{C}_\alpha) = \mathcal{C}$. Thus, measure is *not* a topological invariant.

   (d) Is there a diffeomorphism g from $[0, 1]$ to itself that sends $\mathcal{C}_\alpha$ onto $\mathcal{C}$? You don't have to give a formal proof, just an intuitive argument about how you might make it work (or why it shouldn't).

4. Christina says there can be no problem 4. I had one in mind, but too bad.