1. Consider the following interpretation of the terms “point”, “line”, and “incidence”:

- A **point** is any pair of real numbers \((x, y)\) such that \(x\) and \(y\) are not both zero. That is, any point in \(\mathbb{R}^2\) except the origin.

- A **line** is the set of points \((x, y)\) as above for which there are real numbers \(a\) and \(b\) not both zero such that either
  \[
  (x-a)^2 + (y-b)^2 = (a^2 + b^2) \quad \text{or} \quad ax = by.
  \]
  That is, a **line** is any circle or straight line passing through the origin in \(\mathbb{R}^2\).

- A **point** is incident with a **line** if it satisfies the corresponding equation.

(a) [6 points] Does this define an incidence geometry? That is, do axioms I1, I2, and I3 hold? Fully justify your answer.¹

**Solution:** Throughout, we will use **line** or **point** to refer to the objects as defined in the model, and line, point, or circle to refer to the “usual” Euclidean meanings in \(\mathbb{R}^2\).

First, let’s check I1: Given two points \(P = (x_1, y_1)\) and \(Q = (x_2, y_2)\), we need to confirm that they determine a unique **line**. There are two cases:

First, suppose that \(P\) and \(Q\) are collinear with the origin in \(\mathbb{R}^2\). Either the line \(\overrightarrow{PQ}\) is vertical, in which case \(y_1 = y_2 = 0\) and so \(P\) and \(Q\) lie on the line \(y = 0\) (that is, \(a = 0\) and \(b = 1\)), or \(\overrightarrow{PQ}\) has slope \(m\), in which case the line is \(y = mx\) (that is, \(a = m\) and \(b = 1\)). Thus, if \(\overrightarrow{PQ}\) is of line type, it is not possible to find a circle in \(\mathbb{R}^2\) passing through \(P, Q\) and the origin.

In the other situation, \(P, Q,\) and the origin are not collinear in \(\mathbb{R}^2\), so there is a unique circle containing all three (see footnote). In either case, there is a unique **line** defined by \(P\) and \(Q\).

The next two are easy. Axiom I2 holds, since every line has infinitely many points on it. Axiom I3 also holds easily. For example, the points \((1, 0), (-1, 0),\) and \((0, 1)\) do not lie on the same **line**.

Since axioms I1, I2, and I3 hold, this does define an incidence geometry. In fact, with suitable definitions of between and congruence, all of the other axioms except the continuity and parallel axioms hold.

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¹It might be useful for you to remember that for any three distinct non-collinear points in \(\mathbb{R}^2\), there is a unique circle on which they all lie. Also, C.S. Lewis once said the following line: “The next best thing to being wise oneself is to live in a circle of those that are.”
(b) **4 points** Does this model define a Euclidian, elliptic, or hyperbolic geometry (or none of the these)? Again, fully justify your answer.

**Solution:**

Given a line \( l \) and a point \( P \) not on \( l \), we must determine how many lines passing through \( P \) which are parallel to \( l \).

If \( l \) is of circle type, then there are two possibilities. First, if \( P \) lies on the line \( m \) which is tangent to \( l \) at the origin, lines \( l \) and \( m \) are parallel. There can be no other parallel line (all other circles or lines passing through \( P \) will intersect \( l \)). In the other possibility, there is a unique circle through \( P \) which will be tangent to \( l \) at the origin. In both cases, there is a unique line through \( P \) parallel to \( l \).

In case \( l \) is of line type, then the line in \( \mathbb{R}^2 \) connecting \( P \) to the origin is a line \( m \) parallel to \( l \). But also the circle which passes through \( P \) and is tangent to \( l \) at the origin is another line \( n \) which is parallel to \( l \). Thus, there are two lines parallel to \( l \) through \( P \).

Since we sometimes have two parallels and sometimes have a unique one, the geometry is not hyperbolic, Euclidean, or elliptic.

2. **10 points** Rewrite the following proof, supplying valid reasons for each of the steps.

In any triangle \( \triangle ABC \), if the perpendicular bisectors of two of the sides of \( \triangle ABC \) meet, all three perpendicular bisectors of the three sides meet in a common point \( X \), called the circumcenter.

**Proof:** Let \( M \), \( N \), and \( P \) be the midpoints of sides \( AB \), \( BC \), and \( CA \) respectively. Let the perpendicular bisector of \( AB \) meet that of \( BC \) at a point \( X \). We must show that \( PX \) is perpendicular to \( AC \).

Now, \( \triangle MXB \cong \triangle MXA \) and \( \triangle NXB \cong \triangle NXC \). Thus \( CX \cong AX \), and so \( \triangle CPX \cong \triangle APX \). Therefore \( PX \perp AC \).

**Solution:** Let \( M \), \( N \), and \( P \) be the midpoints of sides \( AB \), \( BC \), and \( CA \) respectively. Such points exist because we have shown that every segment has a unique midpoint.

Erect the perpendicular to \( AB \) at \( M \), and the perpendicular to \( BC \) at \( N \). Such lines exist as a consequence of axiom C4. Let these meet at a point \( X \) (we have taken that two of the bisectors meet as a hypothesis). We must show that \( PX \) is the perpendicular bisector of \( AC \); that is, that \( PX \perp AC \).

Since \( MX \perp AB \), angles \( \angle AMX \) and \( \angle BMX \) are both right, and consequently are congruent. Furthermore, by the definition of midpoint, \( \overline{AM} \cong \overline{BM} \), and \( \overline{MX} \) is congruent to itself. Thus we can apply SAS, to get \( \triangle MXB \cong \triangle MXA \). Consequently \( \overline{AX} \cong \overline{BX} \).
By similar reasoning, we also have $\triangle NXB \cong \triangle NXC$ and thus $\overline{BX} \cong \overline{CX}$.

By transitivity of congruence (axiom C2), we have $\overline{CX} \cong \overline{AX}$. Since we also have $\overline{CP} \cong \overline{AP}$ (definition of midpoint) and $\overline{PX} \cong \overline{PX}$, we can apply SSS to see that $\triangle CPX \cong \triangle APX$.

Thus angles $\angle CPX$ and $\angle APX$ are congruent. Since they are also supplementary angles, we have shown that they must be right angles.

Therefore $\overrightarrow{PX} \perp \overline{AC}$, as desired.

3. 10 points Let $\square ABCD$ be a Saccheri quadrilateral, with right angles at $A$ and $B$, and $\overline{AD} \cong \overline{BC}$ as usual. Also, let $E$ and $F$ be the midpoints of $\overline{AD}$ and $\overline{BC}$ respectively, and let $G$ be the point of intersection of $\overline{EC}$ and $\overline{DF}$. Prove that $\triangle EG \cong \triangle FG$.

**Solution:** Because $\square ABCD$ is Saccheri, we know that the summit angles $\angle EDC$ and $\angle FCD$ are congruent.

Furthermore, since $E$ and $F$ are midpoints of congruent segments, we have $\overline{ED} \cong \overline{FC}$. Observing that $\overline{CD}$ is congruent to itself, we can apply SAS to see that triangles $\triangle EDC$ and $\triangle FCD$ are congruent. This tells us that $\angle CDF \cong \angle ECD$. Now we use the congruence of the summit angles again, together with angle subtraction, to conclude that $\angle EDG \cong \angle FGC$.

Next, notice that $\angle EGD$ and $\angle FGC$ are vertical angles, and hence are congruent. This, together with the previous result and the fact that $\overline{ED} \cong \overline{FC}$, enables us to use AAS to see that $\triangle EGD \cong \triangle FGC$. Consequently, $\overline{EG} \cong \overline{FG}$ as desired.

4. 10 points Prove the Hypotenuse-Leg congruence condition. That is, suppose $\triangle ABC$ and $\triangle PQR$ are right triangles with right angles at $\angle B$ and $\angle Q$. Furthermore, suppose that $\overline{AB} \cong \overline{PQ}$ and $\overline{AC} \cong \overline{PR}$. Show that $\triangle ABC \cong \triangle PQR$. (Hint: an isosceles triangle could be helpful.)

**Solution:** Find a point $D$ on $\overrightarrow{BC}$ so that $C \neq B \neq D$ with $\overline{QR} \cong \overline{BD}$.

Since $\overline{BD} \cong \overline{QR}$, $\angle B \cong \angle Q$, and $\overline{AB} \cong \overline{PQ}$, we have $\triangle ABD \cong \triangle PQR$ by SAS. Thus $\overline{AD} \cong \overline{PR} \cong \overline{AC}$.

Therefore $\triangle ADC$ is isosceles. Since the base angles of an isosceles triangle are congruent, we have $\angle C \cong \angle D \cong \angle R$.

Now we have $\overline{AC} \cong \overline{PR}$, $\angle C \cong \angle R$, and $\overline{BC} \cong \overline{QR}$, we have $\triangle ABC \cong \triangle PQR$ by SAS.
5. **10 points** Prove that the Euclidean parallel postulate holds if and only if the following statement holds:

Let lines $l$ and $m$ be parallel, and let line $t$ be perpendicular to $l$. Then $t$ is also perpendicular to $m$.

**Solution:** Let’s call the property that $l \parallel m$ and $t \perp l \implies t \perp m$ by the name “property S”.

First, let’s show that Euclidean geometry implies property $S$ holds.

Assume $l \parallel m$ and $t \perp l$. Now, since the geometry is Euclidean, $t$ must intersect $m$, since if not, both lines $t$ and $l$ would be parallel to $m$. But since in Euclidean geometry the converse to the alternate interior angle theorem holds, the alternate interior angles formed by $l$, $t$, and $m$ must be equal. Consequently $t \perp m$. So property $S$ holds.

Now we show that if property $S$ holds, the geometry must be Euclidean. One way to do this is suppose that we have a pair of parallel lines $l$ and $m$. Pick two points $P$ and $Q$ on $l$, and erect perpendiculars $t$ and $r$ at these points. By property $S$, $t \perp m$ and $r \perp m$. But since the resulting quadrilateral has four right angles, it is a rectangle. If rectangles exist, the geometry is Euclidean.

There are, of course, many other ways to prove both of these implications.

6. **10 points** Let $D$ be any point interior to triangle $\triangle ABC$. Prove that the measure of $\angle BAC$ is less than the measure of $\angle BDC$.

**Solution:** Extend $CD$ to a point $E$ on $AB$. We have $A \star E \star B$ by the crossbar theorem, and so angle $\angle DEB$ is exterior to $\triangle ACE$. Thus, by the exterior angle theorem, $\angle BAC < \angle DEB$. Also, $\angle BDC$ is exterior to triangle $\triangle BED$, so we have $\angle BAC < \angle BED < \angle BDC$.

Most people instead constructed $\overrightarrow{AD}$, meeting $\overrightarrow{BC}$ at a point $F$. Then they applied the exterior angle theorem to see that $\angle CAD < \angle CDF$ and $\angle BAD < \angle BDF$. Finally, they used angle addition to get the desired conclusion:

$$\angle BAC = \angle CAD + \angle BAD < \angle CDF + \angle BDF = \angle BDC$$

That works, too.