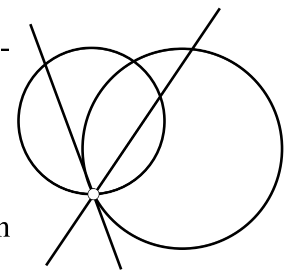
MAT360 Solutions to Midterm

1. Consider the following interpretation of the terms "point", "line", and "incidence":

   - A point is any pair of real numbers (x,y) such that x and y are not both zero. That is, any point in $\mathbb{R}^2$ except the origin.

   - A line is the set of points (x,y) as above for which there are real numbers a and b not both zero such that either

   $$(x-a)^2+(y-b)^2=(a^2+b^2) \qquad \text{or} \qquad ax=by.$$

   That is, a line is any circle or straight line passing through the origin in $\mathbb{R}^2$.

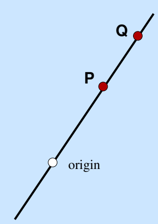
   - A point is incident with a line if it satisfies the corresponding equation.

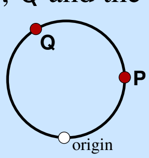
   (a) [6 points] Does this define an incidence geometry? That is, do axioms **I1**, **I2**, and **I3** hold? Fully justify your answer.[1]

   Solution: Throughout, we will use line or point to refer to the objects as defined in the model, and line, point, or circle to refer to the "usual" Euclidean meanings in $\mathbb{R}^2$.

   First, let's check **I1**: Given two points $P=(x_1,y_1)$ and $Q=(x_2,y_2)$, we need to confirm that they determine a unique line. There are two cases:

   First, suppose that P and Q are collinear with the origin in $\mathbb{R}^2$. Either the line $\overleftrightarrow{PQ}$ is vertical, in which case $y_1=y_2=0$ and so P and Q lie on the line $y=0$ (that is, $a=0$ and $b=1$), or $\overleftrightarrow{PQ}$ has slope m, in which case the line is $y=mx$ (that is, $a=m$ and $b=1$). Thus, if $\overleftrightarrow{PQ}$ is of line type, it is not possible to find a circle in $\mathbb{R}^2$ passing through P, Q and the origin.

   In the other situation, P, Q, and the origin are not collinear in $\mathbb{R}^2$, so there is a unique circle containing all three (see footnote). In either case, there is a unique line defined by P and Q.

   The next two are easy. Axiom **I2** holds, since every line has infinitely many points on it. Axiom **I3** also holds easily. For example, the points $(1,0)$, $(-1,0)$, and $(0,1)$ do not lie on the same line.

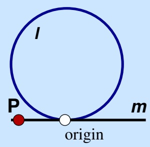
   Since axioms **I1**, **I2**, and **I3** hold, this does define an incidence geometry. In fact, with suitable definitions of between and congruence, all of the other axioms except the continuity and parallel axioms hold.

[1] It might be useful for you to remember that for any three distinct non-collinear points in $\mathbb{R}^2$, there is a unique circle on which they all lie. Also, C.S. Lewis once said the following line: "The next best thing to being wise oneself is to live in a circle of those that are."

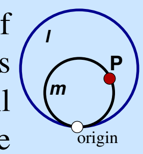
(b) 4 points Does this model define a Euclidian, elliptic, or hyperbolic geometry (or none of the those)? Again, fully justify your answer.

Solution:

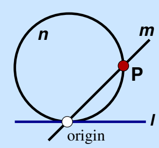
Given a line l and a point P not on l, we must determine how many lines passing through P which are parallel to l.



If l is of circle type, then there are two possibilities. First, if P lies on the line m which is tangent to l at the origin, lines l and m are parallel. There can be no other parallel line (all other circles or lines passing through P will intersect l). In the other possibility, there is a unique circle through P which will be tangent to l at the origin. In both cases, there is a unique line through P parallel to l.



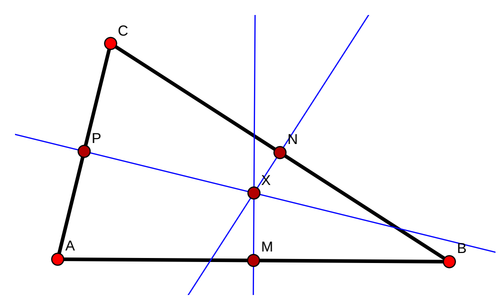
In case l is of line type, then the line in $\mathbb{R}^2$ connecting P to the origin is a line m parallel to l. But also the circle which passes through P and is tangent to l at the origin is another line n which is parallel to l. Thus, there are two lines parallel to l through P.



Since we sometimes have two parallels and sometimes have a unique one, the geometry is not hyperbolic, Euclidean, or elliptic.

2. 10 points Rewrite the following proof, supplying valid reasons for each of the steps.

In any triangle $\triangle ABC$, if the perpendicular bisectors of two of the sides of $\triangle ABC$ meet, all three perpendicular bisectors of the three sides meet in a common point X, called the *circumcenter*.



Proof: Let M, N, and P be the midpoints of sides $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ respectively. Let the perpendicular bisector of $\overline{AB}$ meet that of $\overline{BC}$ at a point X. We must show that $\overline{PX}$ is perpendicular to $\overline{AC}$.
Now, $\triangle MXB \cong \triangle MXA$ and $\triangle NXB \cong \triangle NXC$. Thus $\overline{CX} \cong \overline{AX}$, and so $\triangle CPX \cong \triangle APX$. Therefore $\overleftrightarrow{PX} \perp \overline{AC}$.

Solution: Let M, N, and P be the midpoints of sides $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ respectively. Such points exist because we have shown that every segment has a unique midpoint.

Erect the perpendicular to $\overline{AB}$ at M, and the perpendicular to $\overline{BC}$ at N. Such lines exist as a consequence of axiom **C4**. Let these meet at a point X (we have taken that two of the bisectors meet as a hypothesis). We must show that $\overline{PX}$ is the perpendicular bisector of $\overline{AC}$; that is, that $\overline{PX} \perp \overline{AC}$.

Since $\overline{MX} \perp \overline{AB}$, angles $\angle AMX$ and $\angle BMX$ are both right, and consequently are congruent. Furthermore, by the definition of midpoint, $\overline{AM} \cong \overline{BM}$, and $\overline{MX}$ is congruent to itself. Thus we can apply **SAS**, to get $\triangle MXB \cong \triangle MXA$. Consequently $\overline{AX} \cong \overline{BX}$.

By similar reasoning, we also have $\triangle NXB \cong \triangle NXC$ and thus $\overline{BX} \cong \overline{CX}$.

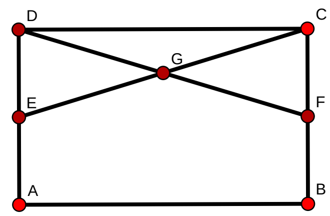
By transitivity of congruence (axiom **C2**), we have $\overline{CX} \cong \overline{AX}$. Since we also have $\overline{CP} \cong \overline{AP}$ (definition of midpoint) and $\overline{PX} \cong \overline{PX}$, we can apply **SSS** to see that $\triangle CPX \cong \triangle APX$.

Thus angles $\angle CPX$ and $\angle APX$ are congruent. Since they are also supplementary angles, we have shown that they must be right angles.

Therefore $\overleftrightarrow{PX} \perp \overline{AC}$, as desired.

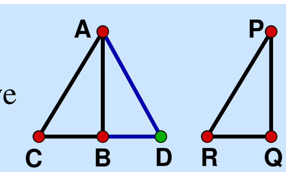
3. [10 points] Let $\square ABCD$ be a Saccheri quadrilateral, with right angles at A and B, and $\overline{AD} \cong \overline{BC}$ as usual. Also, let E and F be the midpoints of $\overline{AD}$ and $\overline{BC}$ respectively, and let G be the point of intersection of $\overline{EC}$ and $\overline{DF}$. Prove that $\overline{EG} \cong \overline{FG}$.

Solution: Because $\square ABCD$ is Saccheri, we know that the summit angles $\angle EDC$ and $\angle FCD$ are congruent.
Furthermore, since E and F are midpoints of congruent segments, we have $\overline{ED} \cong \overline{FC}$. Observing that $\overline{CD}$ is congruent to itself, we can apply **SAS** to see that triangles $\triangle EDC$ and $\triangle FCD$ are congruent.
This tells us that $\angle CDF \cong \angle ECD$. Now we use the congruence of the summit angles again, together with angle subtraction, to conclude that $\angle EDG \cong \angle FCG$.
Next, notice that $\angle EGD$ and $\angle FGC$ are vertical angles, and hence are congruent. This, together with the previous result and the fact that $\overline{ED} \cong \overline{FC}$, enables us to use **AAS** to see that $\triangle EGD \cong \triangle FGC$.
Consequently, $\overline{EG} \cong \overline{FG}$ as desired.



4. [10 points] Prove the Hypotenuse-Leg congruence condition. That is, suppose $\triangle ABC$ and $\triangle PQR$ are right triangles with right angles at $\angle B$ and $\angle Q$. Furthermore, suppose that $\overline{AB} \cong \overline{PQ}$ and $\overline{AC} \cong \overline{PR}$. Show that $\triangle ABC \cong \triangle PQR$. (Hint: an isosceles triangle could be helpful.)

Solution: Find a point D on $\overleftrightarrow{BC}$ so that $C \star B \star D$ with $\overline{QR} \cong \overline{BD}$. Since $\overline{BD} \cong \overline{QR}$, $\angle B \cong \angle Q$, and $\overline{AB} \cong \overline{PQ}$, we have $\triangle ABD \cong \triangle PQR$ by **SAS**. Thus $\overline{AD} \cong \overline{PR} \cong \overline{AC}$.



Therefore $\triangle ADC$ is isosceles. Since the base angles of an isosceles triangle are congruent, we have $\angle C \cong \angle D \cong \angle R$.

Now we have $\overline{AC} \cong \overline{PR}$, $\angle C \cong \angle R$, and $\overline{BC} \cong \overline{QR}$, we have $\triangle ABC \cong \triangle PQR$ by **SAS**.

5. [10 points] Prove that the Euclidean parallel postulate holds if and only if the following statement holds:

Let lines l and m be parallel, and let line t be perpendicular to l. Then t is also perpendicular to m.

Solution: Let's call the property that $l \parallel m$ and $t \perp l \implies t \perp m$ by the name "property **S**".

First, let's show that Euclidean geometry implies property **S** holds.

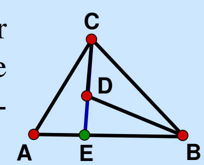
Assume $\overleftrightarrow{l} \parallel \overleftrightarrow{m}$ and $t \perp l$. Now, since the geometry is Euclidean, t must intersect m, since if not, both lines t and l would be parallel to m. But since in Euclidean geometry the converse to the alternate interior angle theorem holds, the alternate interior angles formed by l, t, and m must be equal. Consequently $t \perp m$. So property **S** holds.

Now we show that if property **S** holds, the geometry must be Euclidean. One way to do this is suppose that we have a pair of parallel lines l and m. Pick two points P and Q on l, and erect perpendiculars t and r at these points. By property **S**, $t \perp m$ and $r \perp m$. But since the resulting quadrilateral has four right angles, it is a rectangle. If rectangles exist, the geometry is Euclidean.

There are, of course, many other ways to prove both of these implications.

6. [10 points] Let D be any point interior to triangle $\triangle ABC$. Prove that the measure of $\angle BAC$ is less than the measure of $\angle BDC$.

Solution: Extend $\overline{CD}$ to a point E on $\overline{AB}$. We have $A \star E \star B$ by the crossbar theorem, and so angle $\angle DEB$ is exterior to $\triangle ACE$. Thus, by the exterior angle theorem, $\angle BAC < \angle DEB$. Also, $\angle BDC$ is exterior to triangle $\triangle BED$, so we have $\angle BAC < \angle BED < \angle BDC$.



Most people instead constructed $\overrightarrow{AD}$, meeting $\overline{BC}$ at a point F. Then they applied the exterior angle theorem to see that $\angle CAD < \angle CDF$ and $\angle BAD < \angle BDF$. Finally, they used angle addition to get the desired conclusion:

$$\angle BAC = \angle CAD + \angle BAD < \angle CDF + \angle BDF = \angle BDC$$

That works, too.