1. (Chapter 4,#10)

Prove Proposition 4.7: Hilbert’s Euclidean parallel postulate \(\iff\) If a line intersects one of two parallel lines, it must also intersect the other.

Deduce a corollary that transitivity of parallelism is equivalent to Hilbert’s Euclidean parallel postulate.

**Solution:** First we show that Hilbert’s parallel postulate implies the line crossing condition.

Suppose we have two parallel lines \(l\) and \(m\), and another line \(t\) which intersects \(l\). Let \(P\) be the point where \(l\) and \(t\) intersect. We must show there is a point \(Q\) where \(m\) and \(t\) intersect. But if there is no such \(Q\), then lines \(t\) and \(m\) are parallel. However, by Hilbert’s parallel postulate, there is at most one line which is parallel to \(m\) and contains the given point \(P\). Thus we have a contradiction.

Now let us show the converse: for any pair of parallel lines \(l\) and \(m\), we know that if a line \(t\) crosses one, it must also cross the other. We must establish that Hilbert’s parallel postulate holds under this assumption. So let \(l\) be a given line, and \(P\) be a point not on it; we must show there are not two lines \(m\) and \(n\) passing through \(P\) which are parallel to \(l\). If so, then \(m \parallel l\), and so since \(n\) crosses \(m\) at \(P\), it must cross \(l\). But this contradicts the assumption that \(l\) and \(n\) were parallel.

Now we are to deduce that transitivity of parallelism is equivalent to Hilbert’s parallel postulate. That is,

\[(l \parallel m \text{ and } m \parallel n \Rightarrow l \parallel n) \iff \text{Hilbert’s parallel axiom.}\]

But observe that the contrapositive of transitivity of parallelism is precisely the statement we dealt with before. That is, the contrapositive is

\[l \parallel n \Rightarrow l \parallel m \text{ or } m \parallel n\]

or, in words, “if \(l\) crosses \(n\), then either \(l\) crosses \(m\) or \(n\) crosses \(m\)”. If we use \(t\) instead of \(n\) and assuming \(l \parallel m\), we have the statement above: “if \(l\) crosses \(t\), then \(t\) crosses \(m\)” (since we cannot have \(l\) crossing \(m\)).

2. (Ch. 4, #11) Prove that Hilbert’s parallel postulate is equivalent to the converse of the Alternate Interior Angles theorem.

**Solution:** First, we assume the converse to AIA, and establish Hilbert’s parallel postulate. We have a line \(l\) and a point \(P\), and want to demonstrate that there is at most one line parallel to \(l\) containing \(P\).
Construct a perpendicular $t$ to $l$ that contains $P$, and then let $m$ be the line perpendicular to $t$. For notational purposes, let $A$ be the point where $t$ and $l$ intersect, $B$ be another point $l$, and $C$ be a point of $m$ on the other side of $t$ from $B$. (See the figure).

Now suppose that Hilbert’s parallel postulate fails to hold, that is, there is another line $n$ which contains $P$ and is parallel to $l$. Let $D$ be a point on $n$ on the opposite side of $\overrightarrow{AP}$ from $B$. By the converse of AIA, since $n \parallel l$ and $\angle BAP$ and $\angle APD$ are alternate interior angles, $\angle BAP \cong \angle APD$. But this means that $m = n$, by congruence axiom C4.

For the other direction, we assume Hilbert’s parallel postulate and show that whenever two parallel lines $l$ and $m$ are cut by a transversal $t$, the resulting alternate interior angles are congruent.

Suppose then that we have line $l = \overrightarrow{AB}$ cut by a transversal $t = \overrightarrow{AP}$, with $n = \overrightarrow{PD}$ being parallel to $\overrightarrow{AB}$. Suppose also, for contradiction, that the alternate interior angles $\angle BAP$ and $\angle APD$ are not congruent. Then, by axiom C4, we can create line $\overrightarrow{PC}$ so that $\angle BAP \cong \angle APC$. Applying the Alternate Interior Angle theorem (not the converse!), we know that $\overrightarrow{AB} \parallel \overrightarrow{PC}$. But this contradicts Hilbert’s postulate, since we have two lines containing $P$ that are parallel to $l$.

3. (Ch 4. #14) Fill in the details of Heron’s proof of the triangle inequality (for any triangle $\triangle ABC$, we have $|AB| + |AC| > |BC|$).

**Solution:** Given $\triangle ABC$, bisect $\angle A$, and let the bisector meet $\overrightarrow{BC}$ at a point $D$ (which must exist because of the crossbar theorem).

Observe that $\angle ADC$ is an exterior angle to $\triangle ABD$, so $\angle ADC > \angle BAD$, but $\angle BAD = \angle DAC$ (since we bisected the angle at $A$). Thus, $|AC| > |DC|$ since in any triangle, the greater angle is opposite the longer side.

Similarly, $\angle ADB$ is an exterior angle to $\triangle ADC$, so $\angle ADB > \angle DAC = \angle BAD$, and thus $|AB| > |BD|$. Adding these together gives

$$|AB| + |AC| > |BD| + |DC| = |BC|,$$

as desired.
4. (Ch.4, Major Ex. 5) Given a Saccheri quadrilateral $\square ABCD$ and a point $P$ between $C$ and $D$. Let $Q$ be the foot of the perpendicular from $P$ to the base $AB$. Then show that

(a) $|PQ| < |BD|$ if and only if the summit angles of $\square ABCD$ are acute.

(r) $|PQ| = |BD|$ if and only if the summit angles of $\square ABCD$ are right.

(o) $|PQ| > |BD|$ if and only if the summit angles of $\square ABCD$ are obtuse.

**Solution:** Since $\square ABCD$ is a Saccheri quadrilateral, we know that the summit angles $\angle C$ and $\angle D$ are congruent, and that $AC \cong BD$. Also observe that $\square AQPC$ and $\square QBDP$ are bi-right quadrilaterals; thus we can apply the “greater angle is opposite the greater side” theorem (Prop. 4.13).

Note that angles $\angle CPQ$ and $\angle DPQ$ are supplementary; thus they are either both right angles, or one is acute and one is obtuse. Without loss of generality, we may assume that $\angle CPQ \leq \angle DPQ$. If they are equal, both are right angles, and if not, then we will assume $\angle CPQ$ is acute and $\angle DPQ$ is obtuse.

First, let us establish the forward direction of all three cases:

Suppose $|PQ| < |BD|$. Since $|BD| = |AC|$ by hypothesis, we can apply the “greater angle/longer side” theorem to see that $\angle C < \angle CPQ$. But $\angle CPQ \leq 90^\circ$, and so $\angle C$ must be acute. Hence the summit angles are acute.

Now if $|PQ| > |BD|$, then we know that $\angle D > \angle DPQ$. But since $\angle DPQ \geq 90^\circ$, $\angle D$ is obtuse.

Finally, if $|PQ| = |BD|$, then $\angle D$ and $\angle QPD$ are congruent, as are $\angle C$ and $\angle QPC$. But since $\angle D \cong \angle C$, we know $\angle QPC \cong \angle QPD$. Since these are supplementary angles, they must be right.

Now we establish the reverse direction, which works in much the same way.

If $\angle D$ is acute, then since $\angle QPD$ is not acute, we have $\angle D < \angle QPD$, and so $|PQ| < |BD|$.

If $\angle C$ is obtuse, then since $\angle QPC$ is not obtuse, $\angle C > \angle QPC$, and so $|PQ| > |AC| = |BD|$.

If the summit angles are right, then $\square AQPC$ and $\square QBDP$ are Lambert quadrilaterals, and so by Cor. 3 to Prop. 4.13, $PQ \cong BD$. (In fact, these must all be rectangles).