37. (expires 5/4) Find all the fixed points of the system

\[
\begin{align*}
\dot{x} &= x^2 + y, \\
\dot{y} &= x(y^2 - 1),
\end{align*}
\]

where a “fixed point” is a solution for which both \(x(t)\) and \(y(t)\) are constant. For each of these solutions you find, describe the behavior of the solutions that have initial conditions nearby. You can use Maple to figure out what happens for nearby points, or you can use more mathematical methods.

38. (expires 5/4) Consider the differential equation \(\dot{z}(t) = F(z(t))\), where the vector \(z(t) = (x(t), y(t))\) and the field \(F(x, y) = (-y, x - y)\). Plot a few solutions. What happens to them when \(t \to +\infty\)? Give a “Maple-proof” that this is a general fact for every solution. [A “Maple-proof” is an argument that is rigorous once we accept Maple results as incontrovertibly true.]

39. (expires 5/4) For the equation \(\dot{z} = F(z), z = (x, y)\), with the vector field

\[
F(x, y) = \langle -x(x^4 + y^4) - y, x - y(x^4 + y^4) \rangle,
\]

prove that the origin is an attractor in the future, i.e., every solution verifies

\[
\lim_{t \to +\infty} z(t) = 0.
\]

Note that in this case, while you can (and probably should) use Maple to get an understanding and do calculations, you should format your answer as a regular mathematical proof.

40. (expires 5/4) We will study the Lotke-Volterra predator-prey equations: In a very simple ecosystem, at the time \(t\) (which is expressed, say, in years), there is a population of \(f(t)\) foxes and \(r(t)\) rabbits. The evolution of these quantities obeys the system

\[
\begin{align*}
\dot{f}(t) &= G_f f(t) + E f(t) r(t), \\
\dot{r}(t) &= G_r r(t) - E f(t) r(t);
\end{align*}
\]

where \(G_f\) and \(G_r\) are the growth rates for the foxes and the rabbits, respectively, in the absence of each other. \(E\) is the probability of a fatal encounter between a fox and a rabbit (normalized per number of foxes and rabbits).

First, write some words to explain why these equations make sense. Then, fix \(G_f = 0.4\), \(G_r = 2.4\) (it’s well-known that rabbits have the tendency to reproduce quickly) and \(E = 0.01\). For a few initial conditions of your choice, plot the trajectories in the \((f, r)\)-plane (say, with \(0 \leq f \leq 1000\) and \(0 \leq r \leq 1000\)). For the same initial conditions, plot the actual solutions too (i.e, \(f(t)\) against \(t\), and \(r(t)\) against \(t\)). Write some comments interpreting how the behaviour of the solutions relates to what happens to the two species.

Finally, repeat the same procedure with \(G_f = -1.1\). Things change substantially. Again, what is the “physical” interpretation of this?