

**NOTE:** Each exercise is worth 10 points and can be turned in at any time before its “expiration date”. At the end of the semester, I will expect you to have turned in at least 1/2 of the exercises assigned. If you do more, I will pick your best grades. If you do less, the missing grades will be counted as zeros. Altogether, these will count the same as one project.

Many of these problems will require you to use the help system and/or read the text to figure out what commands you need to use and how to use them.

1. (expires 2/2) If  $p(x)$  is a complex polynomial with real coefficients, it is well known that it can be factored into a product of linear and quadratic terms with real coefficients, or into a product of linear terms only if the coefficients are allowed to be complex.

Use Maple to write  $p(x) = x^5 - 2x^4 - 10x^3 + 20x^2 - 16x + 32$  as a product of *exact* linear and quadratic factors with real coefficients. By exact, I mean you should leave any non-rational factors expressed as radicals; do not approximate terms like  $\sqrt{3}$  as 1.73205, etc.

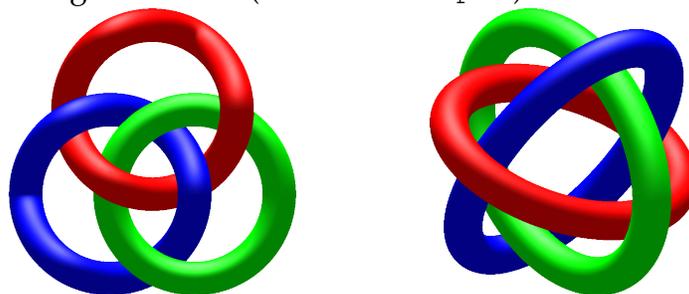
Also write  $p(x)$  as a product of only linear factors.

2. (expires 2/2) Draw a graph showing both  $\cos(x)$  and its fifth Taylor polynomial (that is,  $1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4$ ) for  $x$  between  $-4$  and  $4$ . What degree of Taylor polynomial seems to be needed to get good agreement in this range? *Hint: use a variation of the command `convert(taylor(cos(x), x, 5), polynom)` to make this work. Think of a suitable way to demonstrate that the approximation you have taken is “good” – what is a good definition of “good” here? Be sure to **explain** your choice of the meaning of “good”.*

3. (expires 2/9) The Borromean rings are a set of three mutually interlocked rings, arranged so that if one ring is removed, the other two become unlinked.

Use Maple to make an image of the Borromean rings in  $\mathbb{R}^3$ . *Hint: I suggest using `tubeplot` from the `plots` package. The real challenge of this problem is figuring out how to parameterize the three curves.*

Note that the Borromean rings can not be made from flat, round rings. Either some of the rings have to be bent a little to weave through each other, or flat rings which are elliptical can be used. See the figures below (made with Maple).



4. (expires 2/9) Consider the planar curve  $\gamma$  defined by  $x^2y^3 + y^2 + y - 2e^x = 0$ . Using **only** Maple, find the slope of the tangent line to the curve at  $(0, 1)$ . Then plot the curve and the tangent line on the same graph.

*Hint: you might want to use `implicitplot` from the library `plots`. You might find `implicitdiff` helpful, too.*

5. (expires 2/9) Plot the function  $f(x) = 2 \sin x - x^3 - 1/5$ , for  $x \in [-4, 4]$ . Find all the zeros of the function with an accuracy of 20 decimal digits. *Hint: See `Digits`, `fsolve`.*

6. (expires 2/17) Define a Maple function  $g$  that, given a positive integer  $k$  yields the sum of the first  $k$  primes. What is  $k$  such that  $g(k) \leq 100,000$  but  $g(k+1) > 100,000$ ? You might find `sum` and `ithprime` helpful.
7. (expires 2/21) Use the Maclaurin series for  $\arctan x$  (that is, the Taylor series about  $x = 0$ ) evaluated at  $x = 1/\sqrt{3}$  to compute the value of  $\pi$  to 30 places. How many terms are needed to compute the value to 50 places?
8. (expires 2/17) Fit the points  $(-1.9, -4.7), (-0.8, 1.2), (0.1, 2.8), (1.4, -1.2), (1.8, -3.5)$  by means of a quadratic function  $f(x) = ax^2 + bx + c$ , using the least square method. First, do this step by step, as we did in class; then, use the built-in Maple command, described in the notes. Check that the two solutions agree.

You can load these data points from the web at <http://www.math.sunysb.edu/~scott/mat331.spr12/problems/prob89.txt>, which defines a list `prob8` containing these points.

See the page on [loading a file from the web](#) from the class web page for more details.

9. (expires 2/17) Fit the set of points

$$(1.02, -4.30), (1.00, -2.12), (0.99, 0.52), (1.03, 2.51), (1.00, 3.34), (1.02, 5.30)$$

with a line, using the least square method we used in class. You will see that this is not a good fit. Think of a better way to find a line which *is* a good fit and use Maple to do it. Explain in your solution why you think your better way is indeed better.

You can load these data points from the web at <http://www.math.sunysb.edu/~scott/mat331.spr12/problems/prob89.txt>, which defines a list `prob9` containing these points.

See the page on [loading a file from the web](#) from the class web page for more details.

10. (expires 2/15) The file <http://www.math.sunysb.edu/~scott/mat331.spr12/problems/prob10.txt> defines 21 data points that approximates an exponential curve of the form  $y = ae^{bx}$ . This data is called `prob10`.

Find  $a$  and  $b$  by taking the appropriate logarithm, then use `LeastSquares` to find the resulting “best” line. Transform this line appropriately to get an exponential curve.

Plot the exponential and the `prob10` data on the same axes.

(Since we will be doing exactly this process in class next time, the problem is due abnormally early.)

11. (expires 2/28) In this problem we will estimate the charge of the electron.

If an electron of energy  $E$  is thrown into a magnetic field  $B$  which is perpendicular to its velocity, the electron will be deflected into a circular trajectory of radius  $r$ . The relation between these three quantities is:

$$B r e = \frac{E^2}{m^2 c^4} \sqrt{E^2 - m^2 c^4}, \quad (1)$$

where  $e$  and  $m$  are, respectively, the charge and the mass of the electron, and  $c$  is the speed of light. The rest mass of the electron is defined as  $E_0 = mc^2$ , and is about equal to  $8.817 \cdot 10^{-14}$  Joules. In our experimental set-up the energy of the emitted electrons is set to be  $E = 2.511E_0$ .

The file [electron.txt](#) defines a list called `electron`. Each element of the list is a pair of the form  $[B_i, r_i]$ , and these quantities are expressed in Teslas and meters. Use least square fitting to determine the best value for  $e$ .

[Hint: Notice that the right hand side of eqn (1) is just a constant—calculate it once and for all and give it a name. Then eqn (1) becomes an equation which is linear in the unknown  $e$ . To verify your solution:  $e \approx 1.602 \cdot 10^{-19}$  Coulomb]. Physical constants courtesy of N.I.S.T.

12. (expires 2/28) The set of twelve points

$$\begin{array}{cccc} (-2.256, 0.879) & (-1.764, 5.800) & (-0.684, -0.854) & (-0.776, 6.750) \\ (3.718, 7.394) & (0.081, -1.315) & (-2.357, 4.534) & (6.485, 4.021) \\ (6.518, 3.999) & (2.818, 7.689) & (1.788, -1.668) & (-2.720, 2.719) \end{array}$$

approximate a circle with an unknown radius and center at  $(2, 3)$ . What is the “best” value for  $r$  corresponding to this data? Explain your answer. Plot the resulting circle and the data points on the same graph.

(Note that if you fit the data using the method described in section 6 of the notes, you’ll get a somewhat different radius. Using an unknown center gives one at approximately  $(1.970, 3.002)$  due to the noise in the data.)

If you don’t want to retype the points, you can load them from the file [circcdata.txt](#) on the problems area on the class web page, which defines a list called `data`.

**NOTE:** Neither of the next two problems intrinsically involve `Maple`, except as a word processor to write your solution (although you can use it to help with calculations if you want). If you prefer, you are welcome to turn in your solution as a PDF file, which you can produce with a word processor or by scanning a hand-written version. A worksheet is slightly preferred.

13. (expires 2/28) Following Section 6 (Fitting a Circle) of the notes, prove that if we describe the circle of center  $(a, b)$  and radius  $r$  using the parameters  $(a, b, k)$ , with  $k = a^2 + b^2 - r^2$ , rather than the more natural parameters  $(a, b, r)$ , then the function  $H(a, b, k) = E(a, b, \sqrt{a^2 + b^2 - k})$  is quadratic in  $a, b$  and  $k$ . What does this imply about the number of critical points?
14. (expires 2/28) With reference to the previous problem, show that for  $r > 0$ , the transformation  $(a, b, r) \mapsto (a, b, k)$  is a valid change of variables, that is, it is a diffeomorphism (a one-to-one function with continuous nonzero derivatives). This should help you prove that  $E(a, b, r)$  has only one “physical” critical point, which is a minimum, and is mapped, through the transformation, into the unique critical point of  $H(a, b, k)$ .

15. (expires 3/1) The file [edata.txt](#) on the class website defines a list called `edata`; these points approximate an ellipse of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

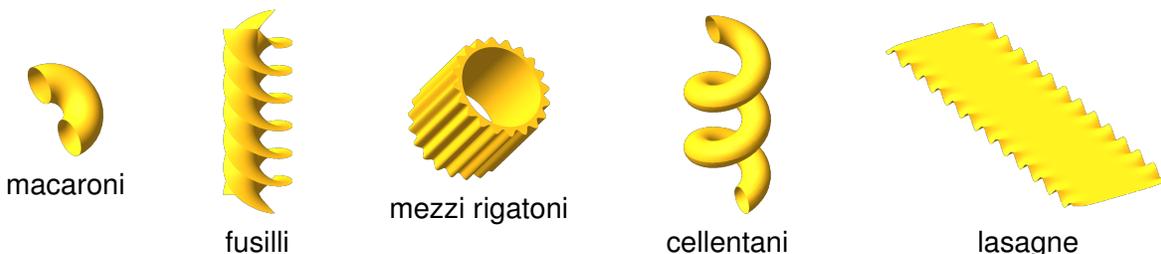
Adapt the method we used in class (and in section 6 of the notes) to find the ellipse that best fits this data.

Then plot both the points in `edata` and the ellipse on the same graph. Make sure the view in your plot is such that your ellipse looks like an ellipse instead of a circle.

You might find it helpful to know that the ellipse above can be described in parametric form as

$$x = a \cos(\theta) \quad y = b \sin(\theta), \quad 0 \leq \theta < 2\pi.$$

16. (expires 3/8) Use `Maple` to make pictures of the following pasta.



Here are some relevant equations, in no particular order.

$$z = \sin(2y) \left( 1 - e^{-(x/6)^8} \right) \quad -6 \leq x \leq 6, \quad -20 \leq y \leq 20$$

$$\tau = 1 \quad 0 \leq \phi \leq \pi, \quad -\pi \leq \sigma \leq \pi \quad (\text{toroidal coordinates})$$

$$x = \left( 1 + \frac{\cos(s)}{2} \right) \cos(t) \quad y = \left( 1 + \frac{\cos(s)}{2} \right) \sin(t) \quad z = 0.4t + \frac{\sin(s)}{2} \quad \begin{array}{l} 0 \leq s \leq 2\pi \\ \frac{\pi}{2} \leq t \leq \frac{11\pi}{2} \end{array}$$

$$\left\{ \begin{array}{l} x = r \sin(t) \quad y = r \cos(t) \quad z = t/2 \\ x = r \sin\left(t + \frac{2\pi}{3}\right) \quad y = r \cos\left(t + \frac{2\pi}{3}\right) \quad z = t/2 \\ x = r \sin\left(t - \frac{2\pi}{3}\right) \quad y = r \cos\left(t - \frac{2\pi}{3}\right) \quad z = t/2 \end{array} \right. \quad \begin{array}{l} 0 \leq r \leq 1 \\ 0 \leq t \leq 4\pi \end{array}$$

$$6 \leq r \leq 7 + \sin(20\theta)/2, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 14 \quad (\text{cylindrical coordinates})$$

To help you get started, the `Maple` worksheet called [pasta.mw](#) draws Mezzi Rigatoni. For full credit, your pasta should look like pasta, with appropriate coloring, viewpoint, smoothness, and lighting. Sauce is optional.

17. (expires 3/15) (a last gasp for curve-fitting problems) The file `bdata.txt` on the class website defines a list called `bdata`. This is a list of 54 points which approximate a line, but several of the points are quite far away from the rest.

Using the robust fitting method discussed in class on February 28, or in chapter 7 of the notes, with the objective functions

$$R(m, b, data) = \sum_{i=1}^n \ln(1 + (mx_i + b - y_i)^2) \quad \text{and} \quad S(m, b, data) = \sum_{i=1}^n 1 - e^{-(mx_i + b - y_i)^2}.$$

Determine the best lines for the data using regular least squares, as well as those given by  $R$  and  $S$  above. Plot the data and the lines you found on the same plot, both in a view that shows all the data points, and in another which restricts the  $y$  values to those for which there is a data point within 10 units of a fit line (ie,  $0 < y < 35$ ).

18. (expires 3/15) Find all the solutions to the differential equation

$$\frac{dx}{dt}(t) = -2x(t), \quad t \in \mathbb{R}.$$

Among them, single out the one for which  $x(0) = 3$ . [Hint: read the help page for `dsolve`, or just do it in your head. It is that easy.]

19. (expires 3/15) Have Maple find analytic solutions to the following system of differential equations,

$$\begin{cases} y''(t) - z(t) = e^t, \\ z'(t) - y(t) = 0, \end{cases}$$

with initial conditions:  $y(0) = 1$ ,  $y'(0) = 0$ ,  $z(0) = k$ . Let us denote the solutions by  $y_k(t)$ ,  $z_k(t)$  (since they depend on the parameter  $k$ ).

For  $k$  taking all integer values from -10 to 10, and  $t \in [-4, 2]$ , plot the functions  $y_k$  in blue, and the functions  $z_k$  in red, all on the same graph. (Yes, you will then have 42 functions plotted on the same graph.) [This is certainly a case when you don't want to retype the functions that Maple finds. You will almost certainly need to read the help page for `dsolve`. I also found `subs`, `unapply`, and `seq` useful.]

20. (expires 3/15) For the functions  $y_k(t)$  and  $z_k(t)$  found in the previous problem, plot the parametric curves  $\varphi_k(t) = [y_k(t), z_k(t)]$  for integer values of  $k$  between  $-5$  and  $5$  and  $-6 < t < 4$  on the same graph. Use the `view` option of plot to only show what lies in the region  $-10 < y < 10$ ,  $-10 < z < 10$ , and use a sequence of colors so that each solution is a different color. [HINT: you might find something like `seq(COLOR(HUE, i/11), i=0..10)` useful for the latter.]

21. (expires 3/15) Find all the fixed points of the system

$$\begin{cases} \dot{x} = x^2 + y, \\ \dot{y} = x(y^2 - 1), \end{cases}$$

where a “fixed point” is a solution for which both  $x(t)$  and  $y(t)$  are constant. For each of these solutions you find, describe the behavior of the solutions that have initial conditions nearby. You can use Maple to figure out what happens for nearby points, or you can use more mathematical methods.

NOTE: The fact that there are various notations for differential equations is purely intentional.

22. (expires 4/20) Write a function to compute the  $n$ -th partial sum of the alternating series

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \dots$$

Specifically, the procedure takes  $n$  as input, and returns the sum of the first  $n$  terms, using a loop.

23. (expires 4/20) Investigate the limit of  $\sin(x)/x$  as  $x \rightarrow 0$ . More specifically, define

$$f(x) = \frac{\sin x}{x}$$

and demonstrate that  $\lim_{x \rightarrow 0} f(x) = 1$  as follows.

Write a function that takes  $\epsilon$  as input, and calculates an integer  $N$  so that  $|f(1/N) - 1| < \epsilon$ , using a while loop.

Specifically, compute  $f(1), f(1/2), f(1/3), \dots$  until  $|f(1/N) - 1| < \epsilon$ , and return  $N$ .

24. (expires 4/27) Calculate the determinant of the square Vandermonde matrix

$$V = \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \alpha_1^3 \\ 1 & \alpha_2 & \alpha_2^2 & \alpha_2^3 \\ 1 & \alpha_3 & \alpha_3^2 & \alpha_3^3 \\ 1 & \alpha_4 & \alpha_4^2 & \alpha_4^3 \end{pmatrix}$$

The determinant of an  $n \times n$  Vandermonde matrix can be expressed as

$$\det(V) = \prod_{1 \leq i < j \leq n} (\alpha_j - \alpha_i)$$

Write a procedure `pdetVandermonde` which takes a list  $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_n]$  and calculates its determinant using the formula above. This can be done using a double loop (note that  $i < j$ ).

Test this procedure with the list  $[1, 2, 3, 4]$  corresponding to the matrix

$$V = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{pmatrix}$$

Given a vector  $\alpha$ , the Maple function `LinearAlgebra[VandermondeMatrix]( $\alpha$ )` will create a Vandermonde matrix as above.

For comparison (not as part of the procedure), convert your list to a vector, then use the command `VandermondeMatrix` to generate the corresponding Vandermonde matrix  $V$ , and compare your answer to the one given by `Determinant(V)`.

25. (expires 4/27) Write a function that takes an integer  $N$  as input, and outputs the first  $N$  terms of the Fibonacci sequence given by

$$F_1 = 1, \quad F_2 = 1, \quad F_n = F_{n-1} + F_{n-2} \text{ for } n > 2.$$

Test this with  $N = 35$ .

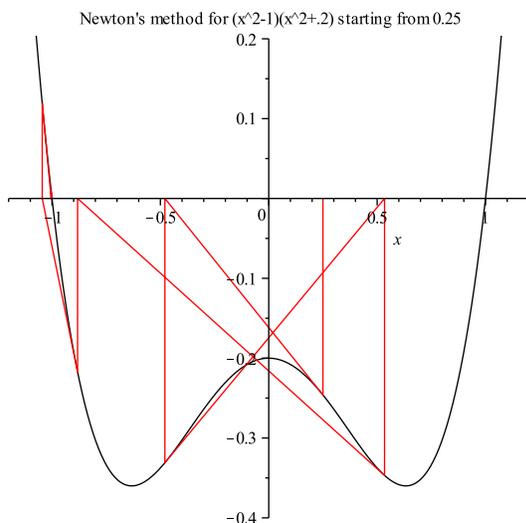
26. (expires 5/10) Write a Maple procedure to implement Newton's method. As input, it should take as input a function  $f$ , an initial guess  $x_0$ , and a number of iterates  $n$ . Then it should calculate

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

optionally printing out  $i$ ,  $x_i$ , and  $f(x_i)$ , as  $i$  ranges from 1 to  $n$ . It should return  $x_n$ .

27. (expires 5/10) Adapt your answer to the previous problem to graphically illustrate the process of Newton's method for any given function and initial guess.

Specifically, your procedure should take a function as input a function  $f$ , an initial value  $x_0$ , a number of iterates  $n$ , and an  $x$ -range for the plot. Then it should print out the values of  $i$ ,  $x_i$  and  $f(x_i)$  for  $0 \leq i \leq n$  (where  $x_{i+1}$  is given by one step of Newton's method from  $x_i$ ), and produce a graph of  $f(x)$ , together with the  $x_i$  indicated as in the graph below.



```

1, 0.25, -0.24609375
2, -0.4791666667, -0.3309640767
3, 0.5341996093, -0.3468597849
4, -0.8818848457, -0.2173269360
5, -1.044991129, 0.1188729407
6, -1.003895062, 0.009427277846
7, -1.000032517, 0.00007804622955
8, -1.000000002, 4.800000016*10-9
    
```

28. (expires 5/10) Consider the following function on positive integers  $n$ . If  $n$  is even, then let  $c(n) = n/2$ . If  $n$  is odd, let  $c(n) = 3n + 1$ .

The Collatz conjecture, proposed in 1937 by Luthar Collatz, is that if for any initial  $n$ ,  $c(n)$  is applied repeatedly, the process eventually arrives at with  $n = 1$ . For example, if we begin with 11, we get the sequence

11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1

and it takes 14 steps to arrive at 1. The sequence beginning with 27 takes 111 steps to arrive at 1.

Write a *recursive* Maple procedure `Collatz` which, given an initial  $n$ , calls itself to determine how many steps are required to arrive at 1. Along the way, it should print out the value of the sequence. For example, `Collatz(3)` should print out the value 3 and return `1+Collatz(10)`. `Collatz(1)` should print 1 and return 0.