MAT331 homework problems

17. (expires 18 March) Consider the differential equation $\dot{z}(t) = F(z(t))$, where the vector $z(t) = (x(t), y(t))$ and the field $F(x, y) = (-y, x - y)$. Plot a few solutions. What happens to them when $t \to +\infty$? Give a "Maple-proof" that this is a general fact for every solution. [A "Maple-proof" is an argument that is rigorous once we accept Maple results as incontrovertibly true.]

18. (expires 18 March) (No Maple.) For the equation $\dot{z} = F(z)$, $z = (x, y)$, with the vector field

$$F(x, y) = \langle -x(x^4 + y^4) - y, x - y(x^4 + y^4) \rangle,$$

prove that the origin is an attractor in the future, i.e., every solution verifies

$$\lim_{t \to +\infty} z(t) = 0.$$

[You can ask around how to do this, but then you have to show clearly that you have understood it.]

19. (expires 18 March) We will study the Lotke-Volterra predator-prey equations: In a very simple ecosystem, at the time $t$ (which is expressed, say, in years), there is a population of $f(t)$ foxes and $r(t)$ rabbits. The evolution of these quantities obeys the system

$$\begin{align*}
\dot{f}(t) &= G_f f(t) + E f(t) r(t), \\
\dot{r}(t) &= G_r r(t) - E f(t) r(t);
\end{align*}$$

where $G_f$ and $G_r$ are the growth rates for the foxes and the rabbits, respectively, in the absence of each other. $E$ is the probability of a fatal encounter between a fox and a rabbit (normalized per number of foxes and rabbits).

First, write some words to explain why these equations make sense. Then, fix $G_f = 0.4$, $G_r = 2.4$ (it’s notorious that rabbits have the tendency to reproduce quickly) and $E = 0.01$. For a few initial conditions of your choice, plot the trajectories in the $(f, r)$-plane (say, with $0 \leq f \leq 1000$ and $0 \leq r \leq 1000$). For the same initial conditions, plot the actual solutions too (i.e, $f(t)$ against $t$, and $r(t)$ against $t$). Write some comments interpreting how the behaviour of the solutions relates to what happens to the two species.

Finally, repeat the same procedure with $G_f = -1.1$. Things change substantially. Again, what is the “physical” interpretation of this?