2010-10-22

Let's apply the fixed point analysis we did last time to the phugoid

> $R := 'R'$:

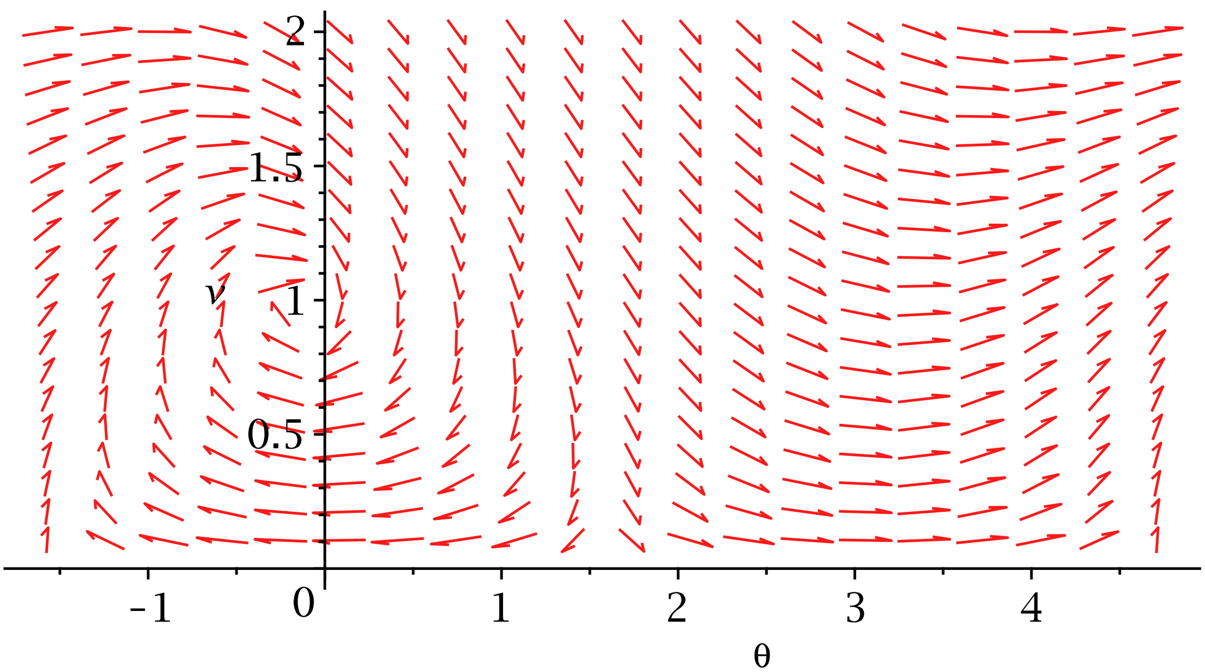
$phug := \left[D(\text{theta})(t) = v(t) - \frac{\cos(\text{theta}(t))}{v(t)}, D(v)(t) = -\sin(\text{theta}(t)) - R \cdot v(t)^2\right]$:

> $with(DEtools)$:

$R := 0.2$;

$DEplot\left(phug, [\text{theta}, v], t = 0..10, \text{theta} = -\frac{\text{Pi}}{2}..\frac{3 \cdot \text{Pi}}{2}, v = 0..2\right)$

$$R := 0.2$$



OK, it seems to be right. Let's put R back to be a variable.

> $R := 'R'$:

I don't want to retype, but I want to solve the right=hand side of both equations = 0 in terms of R. How?

one idea:

> $subs(\ \{D(\text{theta})(t) = 0, D(v)(t) = 0\}, phug)$

$$\left[0 = v(t) - \frac{\cos(\theta(t))}{v(t)}, 0 = -\sin(\theta(t)) - R\, v(t)^2\right] \quad \textbf{(1)}$$

Another way: as Marc suggests, use rhs to extract RightHandSide

> $rhs(this = that)$

$$that \quad \textbf{(2)}$$

> $map(rhs, phug)$

$$\left[v(t) - \frac{\cos(\theta(t))}{v(t)}, -\sin(\theta(t)) - R\, v(t)^2\right] \quad \textbf{(3)}$$

> $vfield := subs(\{v(t) = v, \text{theta}(t) = \text{theta}\}, \textbf{(3)})$

$$vfield := \left[v - \frac{\cos(\theta)}{v}, -\sin(\theta) - R\,v^2\right] \quad \textbf{(4)}$$

>

note that using **(4)** is different from %, since % means the last result, and **(4)** means result with eqn number **(4)**, no matter what order you do things. to get **(4)**, double click on the label. Or just give your result a name, like *vfield*.

> $solve([vfield[1] = 0, vfield[2] = 0])$

$$\left\{R = \frac{RootOf(v^4 + \_Z^2 - 1)}{v^2}, \theta = \arctan(-RootOf(v^4 + \_Z^2 - 1), v^2), v = v\right\} \quad \textbf{(5)}$$

Not what I want, but close. Maple guessed I want a solution in terms of R and theta in terms of v instead of (theta,v) in terms of R. Maybe I should tell it what to solve for.

> $solve([vfield[1] = 0, vfield[2] = 0], \{theta, v\})$

$$\{\theta = \arctan(-RootOf(-1 + (R^2+1)\,\_Z^2)\,R, RootOf(-1 + (R^2+1)\,\_Z^2)), v = RootOf(\_Z^2 - RootOf(-1 + (R^2+1)\,\_Z^2))\} \quad \textbf{(6)}$$

> $convert(\%, radical)$

$$\left\{\theta = \arctan\left(-\sqrt{\frac{1}{R^2+1}}\,R, \sqrt{\frac{1}{R^2+1}}\right), v = \left(\frac{1}{R^2+1}\right)^{1/4}\right\} \quad \textbf{(7)}$$

Why is arctan weird? Cuz it doesn't know if R is complex, or even non negative.

> $solve([vfield[1] = 0, vfield[2] = 0], \{theta, v\})$:
$fixsol := convert(\%, radical)$ assuming $R \geq 0$

$$fixsol := \left\{\theta = -\arctan(R), v = \left(\frac{1}{R^2+1}\right)^{1/4}\right\} \quad \textbf{(8)}$$

> $subs(fixsol, [theta, v])$

$$\left[-\arctan(R), \left(\frac{1}{R^2+1}\right)^{1/4}\right] \quad \textbf{(9)}$$

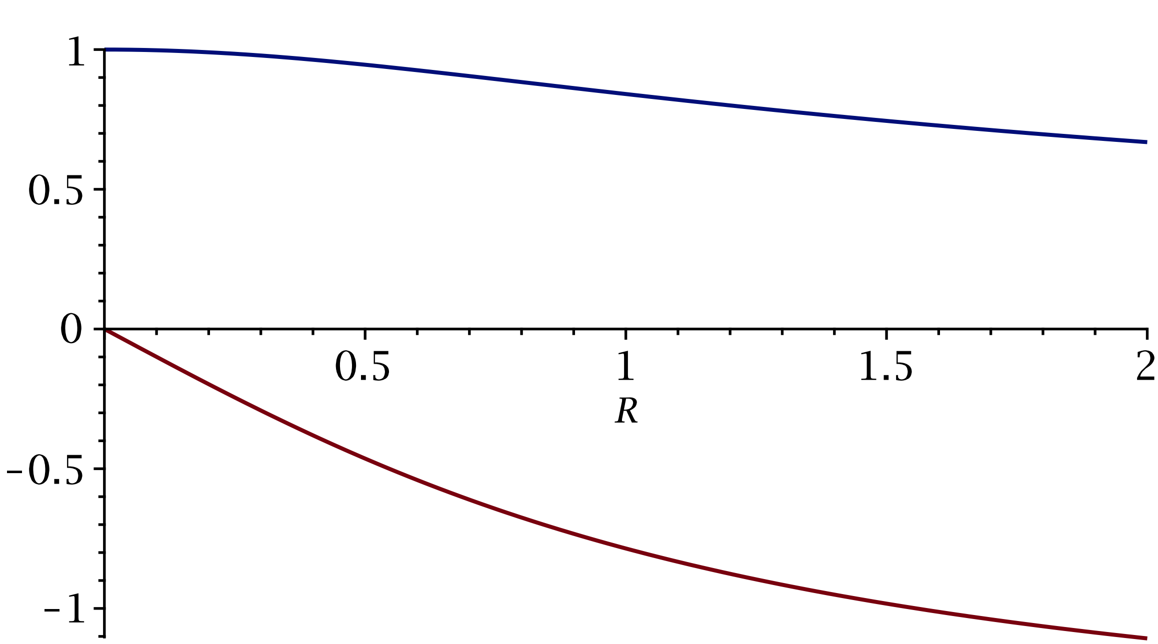
> $Fixpt := unapply(subs(fixsol, [theta, v]), R)$

$$Fixpt := R \mapsto \left[-\arctan(R), \left(\frac{1}{R^2+1}\right)^{1/4}\right] \quad \textbf{(10)}$$

> $Fixpt(.2)$

$$[-0.1973955598, 0.9902427357] \quad \textbf{(11)}$$

> $plot(Fixpt(R), R = 0..2)$



not what I want.
I want [theta, v, R=0..2]

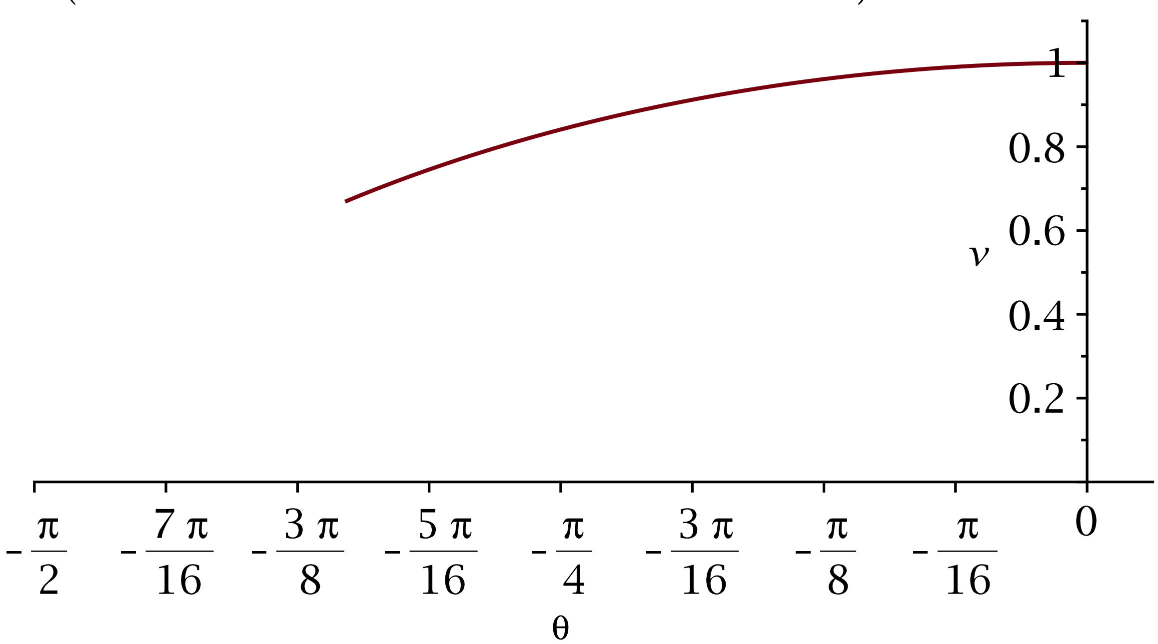
> $op(Fixpt(R))$

$$-\arctan(R), \left(\frac{1}{R^2+1}\right)^{1/4} \quad \textbf{(12)}$$

> $[op(Fixpt(R)), R=0..2]$

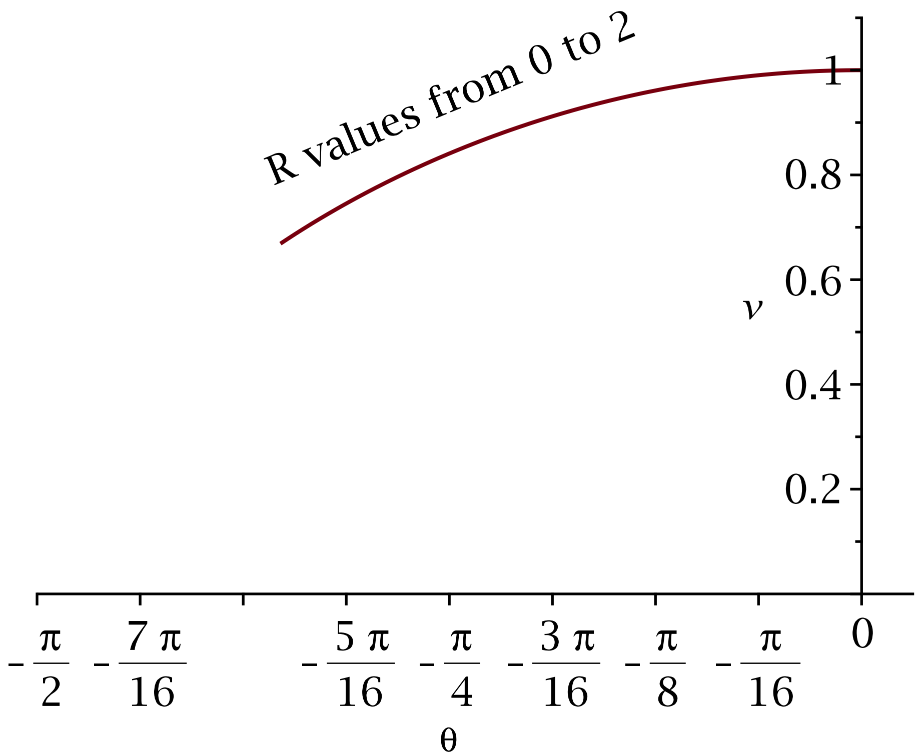
$$\left[-\arctan(R), \left(\frac{1}{R^2+1}\right)^{1/4}, R=0..2\right] \quad \textbf{(13)}$$

> $plot\left([op(Fixpt(R)), R=0..2], \text{theta}=-\frac{\text{Pi}}{2}..0.1, v=0..1.1\right)$



> $with(plots):$

> $display\left(\left[plot\left([op(Fixpt(R)), R = 0..2], \text{theta} = -\frac{\text{Pi}}{2}..0.1, v = 0..1.1, scaling = constrained\right), textplot\left(\left[-\frac{\text{Pi}}{8}, .75, \text{"R values from 0 to 2"}, \right], align = [above, left], rotation = \frac{\text{Pi}}{8}\right)\right]\right)$



> $Fixpt(.4)$

$$[-0.3805063771, 0.9635749534] \quad \textbf{(14)}$$

Goal: Given R, find fixed point (done), and determine its type (elliptic, sink, source, saddle, blah blah) by looking at the Trace and Determinant of the Jacobian.

> $vfield$

$$\left[v - \frac{\cos(\theta)}{v}, -\sin(\theta) - R v^2\right] \quad \textbf{(15)}$$

> $Jacobian$

Jacobian **(16)**

Have to load Jacobian from VectorCalculus package.

> $with(VectorCalculus):$

> $jack := Jacobian(vfield, [\text{theta}, v])$

(17)

$$jack := \begin{bmatrix} \frac{\sin(\theta)}{\nu} & 1 + \frac{\cos(\theta)}{\nu^2} \\ -\cos(\theta) & -2\,R\,\nu \end{bmatrix} \quad \textbf{(17)}$$

>

What is the Jacobian at the fixed point when R=0.4?
A dumb way (but right):

> $JacksADullBoy := eval(jack, \{R = 0.4, \text{theta} = -0.3805063771, \nu = 0.9635749534\})$

$$JacksADullBoy := \begin{bmatrix} -0.3854299813 & 2.000000000 \\ -0.9284766909 & -0.7708599628 \end{bmatrix} \quad \textbf{(18)}$$

To do linear algebra, it is useful to load LinearAlgebra

> $with(LinearAlgebra):$

> $Trace(JacksADullBoy), Determinant(JacksADullBoy)$

$$-1.156289944,\ 2.154065923 \quad \textbf{(19)}$$

> $Trace(JacksADullBoy)^2 < 4 \cdot Determinant(JacksADullBoy)$

$$1.337006435 < 8.616263692 \quad \textbf{(20)}$$

So this is a spiral sink, because
Trace<0, Trace^2 < 4*Det

Lets write the Jacobian at the fixed point in terms of R only. The answer as a function is ugly stuff, so I supress it with :, then evaluate it prettier.

> $JackFix := unapply(\ eval(jack, \{\nu = Fixpt(R)[2], \text{theta} = Fixpt(R)[1]\}), R):$
$JackFix(R)$

$$\begin{bmatrix} -\frac{R}{\sqrt{R^2+1}\left(\frac{1}{R^2+1}\right)^{1/4}} & 1 + \frac{1}{\sqrt{R^2+1}\sqrt{\frac{1}{R^2+1}}} \\ -\frac{1}{\sqrt{R^2+1}} & -2\,R\left(\frac{1}{R^2+1}\right)^{1/4} \end{bmatrix} \quad \textbf{(21)}$$

> $JackFix(0.4)$

$$\begin{bmatrix} -0.3854299817 & 2.000000000 \\ -0.9284766913 & -0.7708599628 \end{bmatrix} \quad \textbf{(22)}$$

> $JackFix(3)$

$$\begin{bmatrix} -\frac{3\,10^{3/4}}{10} & 2 \\ -\frac{\sqrt{10}}{10} & -\frac{3\,10^{3/4}}{5} \end{bmatrix} \quad \textbf{(23)}$$

> $Eigenvalues(JackFix(0.4))$

(24)

$$\begin{bmatrix} -0.578144972250000 + 1.34900493513453\,I \\ -0.578144972250000 - 1.34900493513453\,I \end{bmatrix} \quad (24)$$

Another way to see that we have a spiral sink: complex eigenvalues, with negative real part.
For a big value of R, we have real eigenvalues that are both negative

> *Eigenvalues*(*JackFix*(3.0))

$$\begin{bmatrix} -2.24936529870151 + 0.\,I \\ -2.81170662829849 + 0.\,I \end{bmatrix} \quad (25)$$

>

We will see next time that at $R = 2\sqrt{2}$, we get two negative but equal eigenvalues.
This is actually easy to do, so let me just put it here.

We want to solve where Trace^2 = 4*Det in terms of R.

> *solve*(*Trace*(*JackFix*(*R*))^2 = 4· *Determinant*(*JackFix*(*R*)))

$$2\sqrt{2},\ -2\sqrt{2} \quad (26)$$

> *Eigenvalues*(*JackFix*(2·sqrt(2)))

$$\begin{bmatrix} -\sqrt{2}\,\sqrt{3} \\ -\sqrt{2}\,\sqrt{3} \end{bmatrix} \quad (27)$$