10/10/19
Phugoid continues.

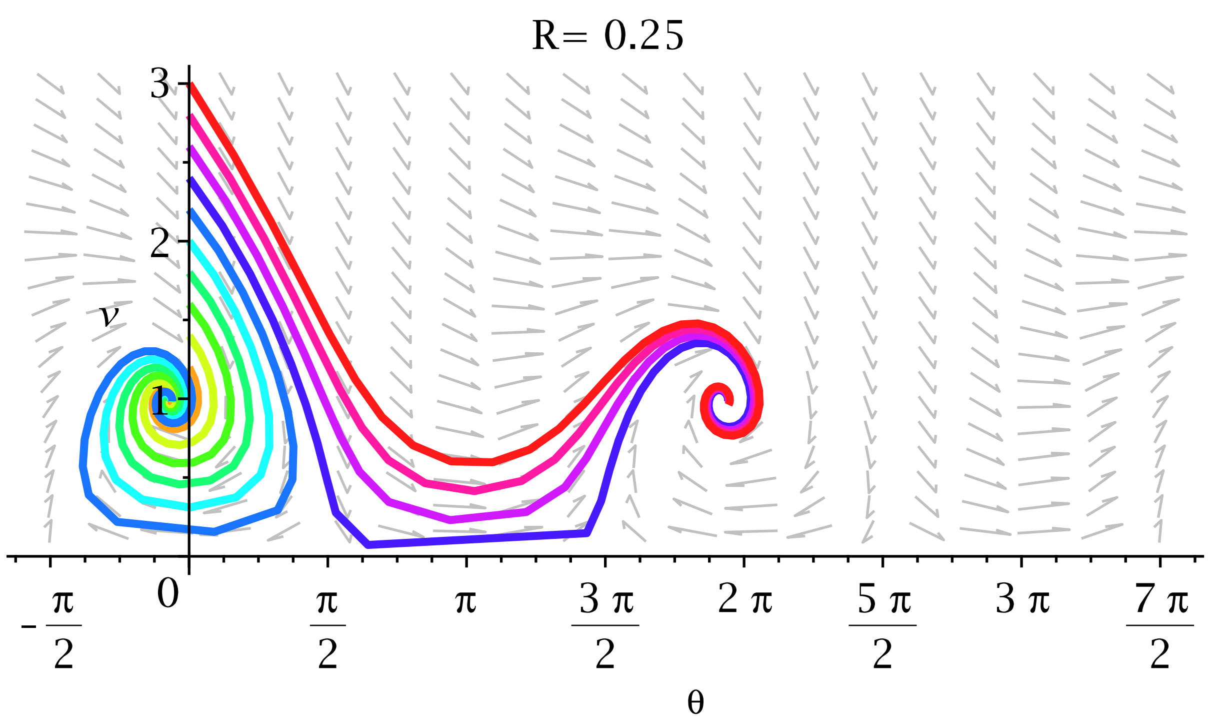
$$\left\{ \frac{d}{dt}\theta(t) = v(t) - \frac{\cos(\theta(t))}{v(t)} \quad , \quad \frac{d}{dt}v(t) = -\sin(\theta(t)) - R\cdot v(t)^2 \right\}$$

Let's look a a solution to the phugoid, with R=.25

> $phug := \left[\mathrm{D}(\mathrm{theta})(t) = v(t) - \frac{\cos(\mathrm{theta}(t))}{v(t)}, \mathrm{D}(v)(t) = -\sin(\mathrm{theta}(t)) - R \cdot v(t)^2 \right]:$

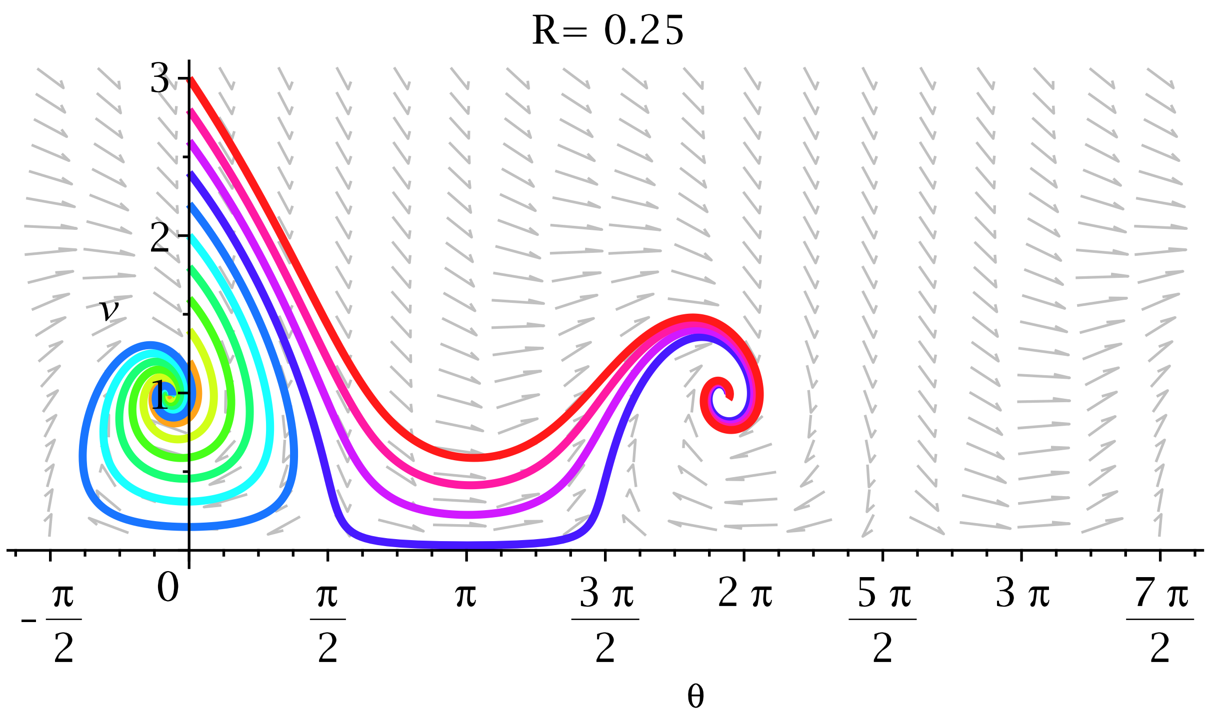
> $R := .25:$
> $inits := [seq([\mathrm{theta}(0) = 0, v(0) = initv],\ initv = 1..3, .2)\ ]:$
> \# *these are my initial values...*
> $with(DEtools):$

> $DEplot\Big(\ phug, [\mathrm{theta}(t), v(t)], t = 0..10,\ inits,$
> $\mathrm{theta} = -\frac{\mathrm{Pi}}{2}..\frac{7\,\mathrm{Pi}}{2}, v = 0..3, tickmarks = [piticks, default],$
> $linecolor = [seq(COLOR(HUE, hval), hval = 0..1, 0.1)], color = gray,$
> $title = sprintf(\text{"R=\%5.2f"}, R)\Big)$



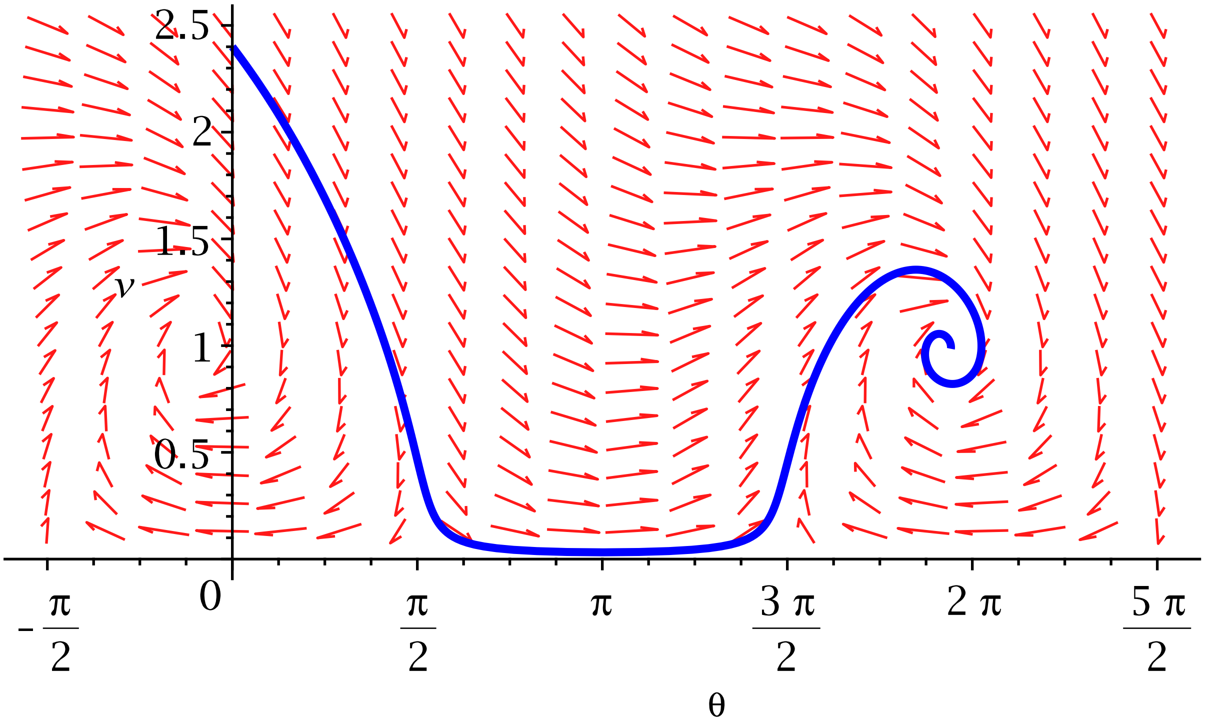
This is not "believable" for the **dark blue** solution... the region from about $\pi/2$ to $3\pi/2$ is too straight. Why?
This can be fixed by adjusting the stepsize (but I haven't said what that means):

> $DEplot\left(phug, [\text{theta}(t), v(t)], t=0..10, inits, \text{theta} = -\frac{\text{Pi}}{2} .. \frac{7\,\text{Pi}}{2}, v = 0..3, tickmarks = [piticks, default], linecolor = [seq(COLOR(HUE, hval), hval = 0..1, 0.1)], color = gray, stepsize = .01, title = sprintf(\text{"R=\%5.2f"}, R) \right)$



Let's just look at one solution.

> $DEplot\left(phug, [\text{theta}(t), v(t)], t=0..10, [[v(0) = 2.4, \text{theta}(0) = 0]], \text{theta} = -\frac{\text{Pi}}{2} .. \frac{5\,\text{Pi}}{2}, v = 0..2.5, linecolor = blue, stepsize = .01, tickmarks = [piticks, default] \right)$



This is a picture of the solution in the (θ,v) plane. Can we plot graphs of θ(t) (as a function of t) and of v(t)?

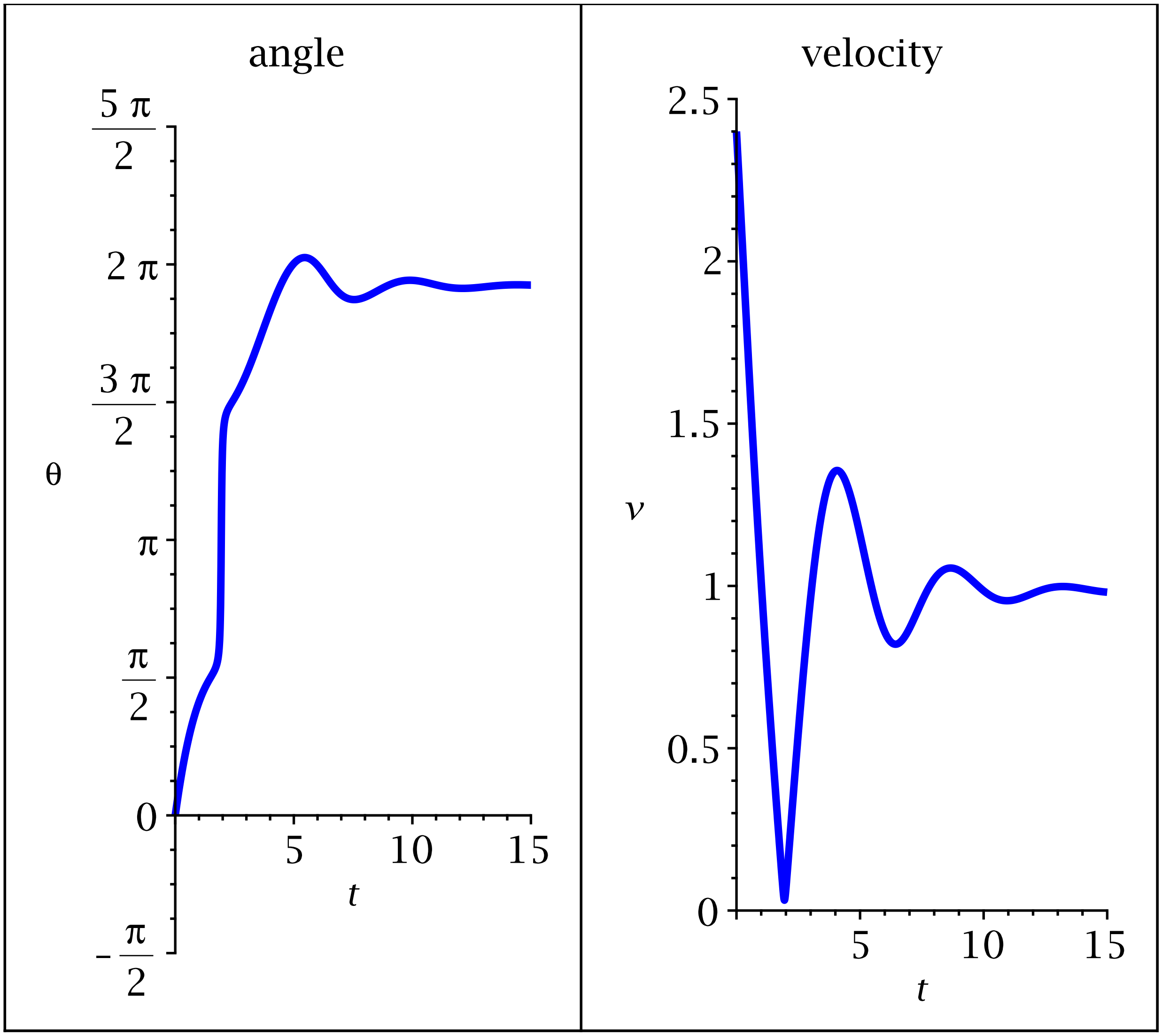
To do this, we use the **scene** option of **DEplot**.

As I've done before, I'm going to use **display(< pic1 | pic2 >)** to show two images side-by-side.

$$> \ with(plots):$$

$$display\left(\left\langle DEplot\left(phug, [\theta(t), v(t)], t = 0..15, [[v(0) = 2.4, \theta(0) = 0]], \theta = -\frac{Pi}{2}..\frac{5\,Pi}{2}, v = 0..2.5, linecolor = blue, stepsize = .01, scene = [t, \theta], title = \text{"angle"}, tickmarks = [default, piticks]\right) \right| \right.$$

$$\left. DEplot\left(phug, [\theta(t), v(t)], t = 0..15, [[v(0) = 2.4, \theta(0) = 0]], \theta = -\frac{Pi}{2}..\frac{5\,Pi}{2}, v = 0..2.5, linecolor = blue, stepsize = .01, scene = [t, v], title = \text{"velocity"}\right) \right\rangle\Big)$$

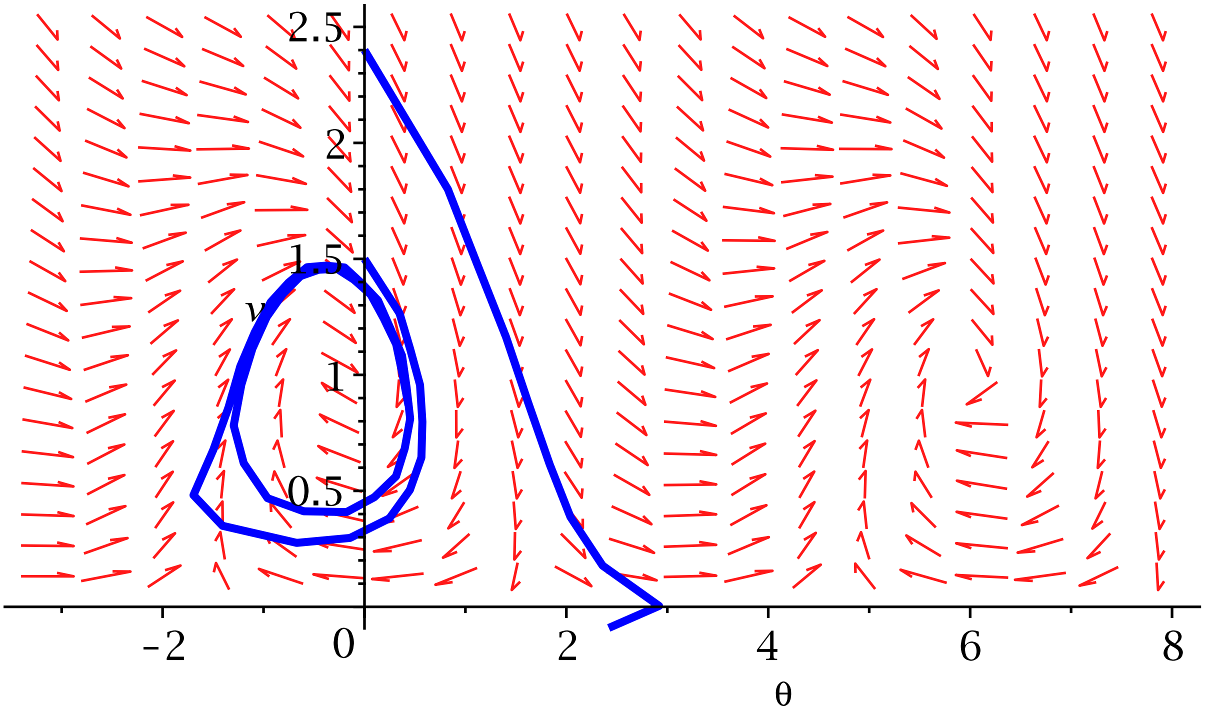


Looking at the (t,θ) graph, observe that θ grows almost linearly for 0<t<2, but at about t=2, it changes very rapidly. Then for t>0, θ increases more slowly, levels off at about t=5, and oscillates slightly, limiting on an angle just below 2π.
By comparison, the (t,v) graph shows the velocity drops rapidly to about t=2, the abruptly reverses direction, increasing until about t=4, where it decreases again, oscillating towards a limiting value of about 1.

Let's return to the issue of the stepsize.

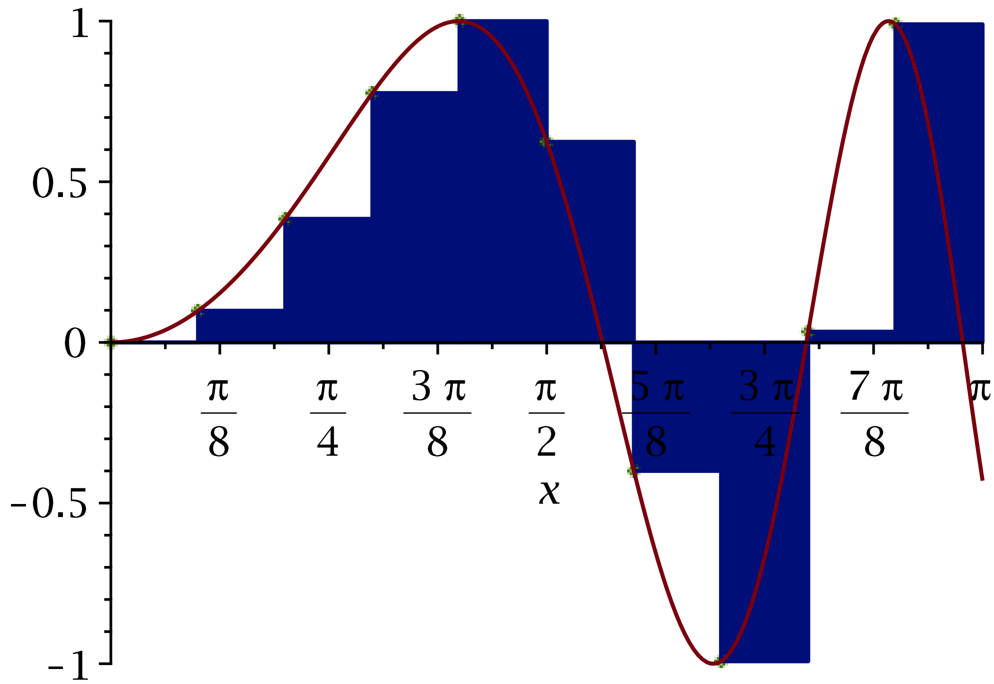
First, there was some discussion of Euler's method, and an illustration here. Using Euler's method with a stepsize of .4 gives us some pretty terrible results.

> $DEplot\left(phug, [\mathrm{theta}(t), v(t)], t = 0..10, [[v(0) = 1.5, \mathrm{theta}(0) = 0], [v(0) = 2.4, \mathrm{theta}(0) = 0]], \mathrm{theta} = -\mathrm{Pi}..\frac{5\,\mathrm{Pi}}{2}, v = 0..2.5, linecolor = blue, stepsize = .4, method = classical[foreuler]\right)$



There was a longish discussion about relation between Euler's method, Right-hand Riemann sums, Trapezoid rule, Simpson's rule and Runge-Kutta; I won't repeat it here. Read the text.

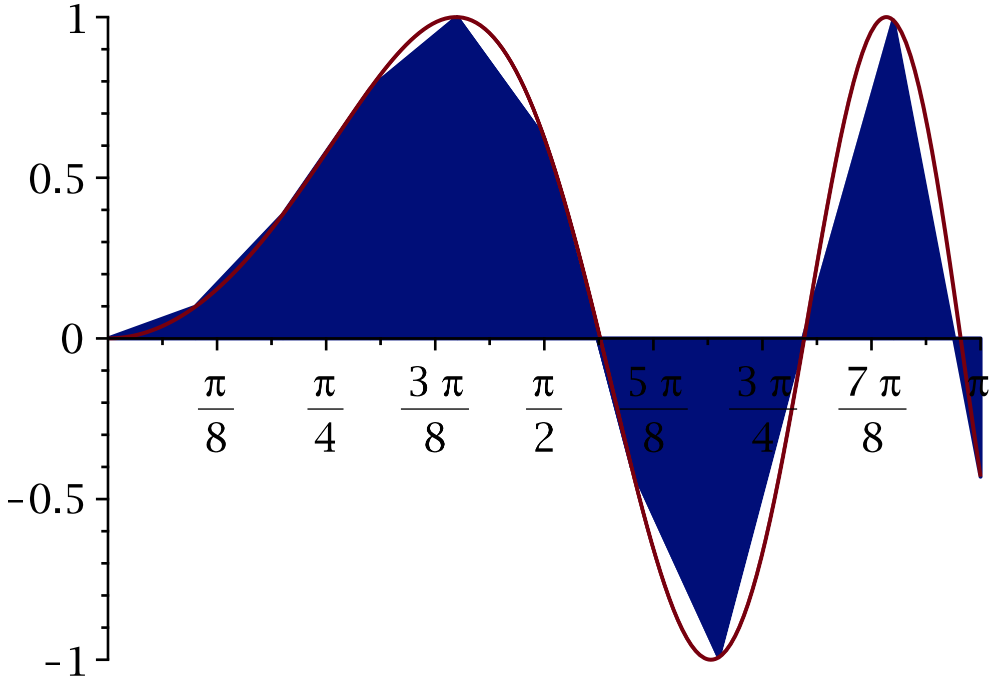
```
> with(Student[Calculus1]) :
> ApproximateIntTutor( )
> RiemannSum(sin(x^2), 0 .. Pi, 'partition' = 10, 'method' = left, 'partitiontype'
    = normal, 'output' = 'plot', 'boxoptions' = ['filled' = ['transparency'= .5]]);
```



A left Riemann sum approximation of $\int_0^{\pi} f(x)\,\mathrm{d}x$,

where $f(x) = \sin\left(x^2\right)$ and the partition is uniform.
The approximate value of the integral is
0.7899725452. Number of subintervals used: 10.

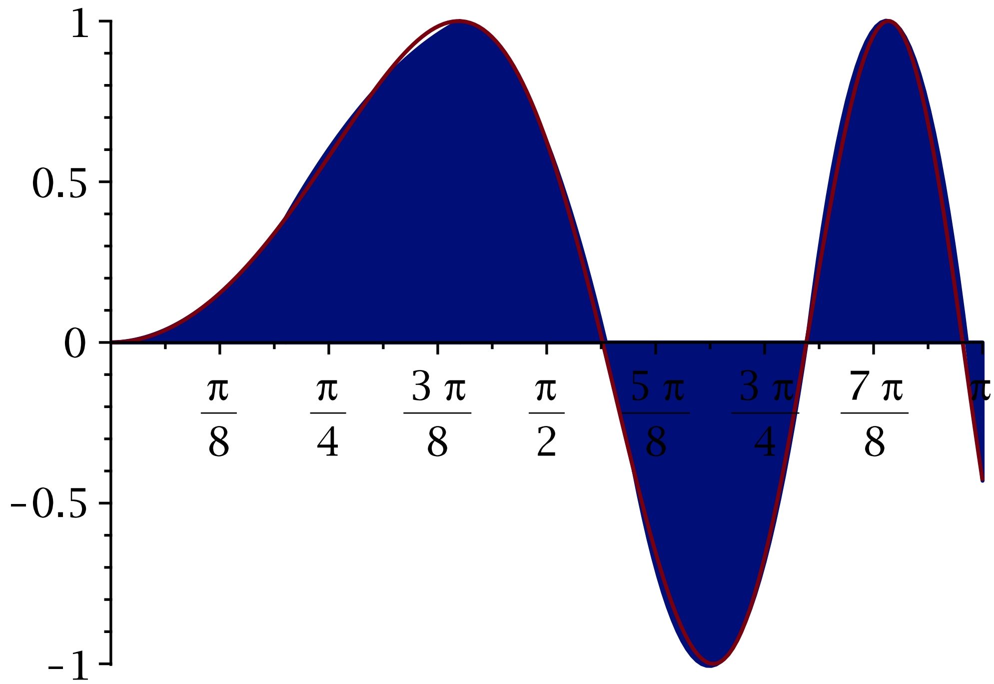
> *ApproximateInt*(sin(*x*^2), 0 .. Pi, '*partition*' = 10, '*method*' = *trapezoid*, '*partitiontype*' = *normal*, '*output*' = '*plot*', '*boxoptions*' = ['*filled*' = ['*transparency*'= .5]]);



An approximation of $\int_0^{\pi} f(x)\,\mathrm{d}x$ using trapezoid rule,

where $f(x) = \sin(x^2)$ and the partition is uniform.
The approximate value of the integral is
0.7223809884. Number of subintervals used: 10.

> *ApproximateInt*(sin(*x*^2), 0 .. Pi, '*partition*' = 10, '*method*' = *simpson*, '*partitiontype*' = *normal*, '*output*' = '*plot*', '*boxoptions*' = ['*filled*' = ['*transparency*'= .5]]);



An approximation of $\int_0^{\pi} f(x)\,\mathrm{d}x$ using Simpson's rule,
where $f(x) = \sin(x^2)$ and the partition is uniform.
The approximate value of the integral is
0.7950310646. Number of subintervals used: 5.

Now that we understand more about numerical integration (both for integrals and differential equations), let's return to the Phugoid model. How can we plot the motion of the plane through the air, that is, in terms of x(t), y(t) ?

Observe that $\frac{dx}{dt} = v \cdot \cos(\theta)$, $\frac{dy}{dt} = v \cdot \sin(\theta)$

This means we can augment our system to get a vector field in $\mathbb{R}^4$, where we have (θ, v,x,y) evolving with time.

> $xphug := \left[\mathrm{D}(\mathrm{theta})(t) = v(t) - \frac{\cos(\mathrm{theta}(t))}{v(t)}, \mathrm{D}(v)(t) = -\sin(\mathrm{theta}(t)) - R \cdot v(t)^2, \right.$
>
> $\left. \mathrm{D}(x)(t) = v(t) \cdot \cos(\mathrm{theta}(t)), \quad \mathrm{D}(y)(t) = v(t) \cdot \sin(\mathrm{theta}(t)) \right]:$

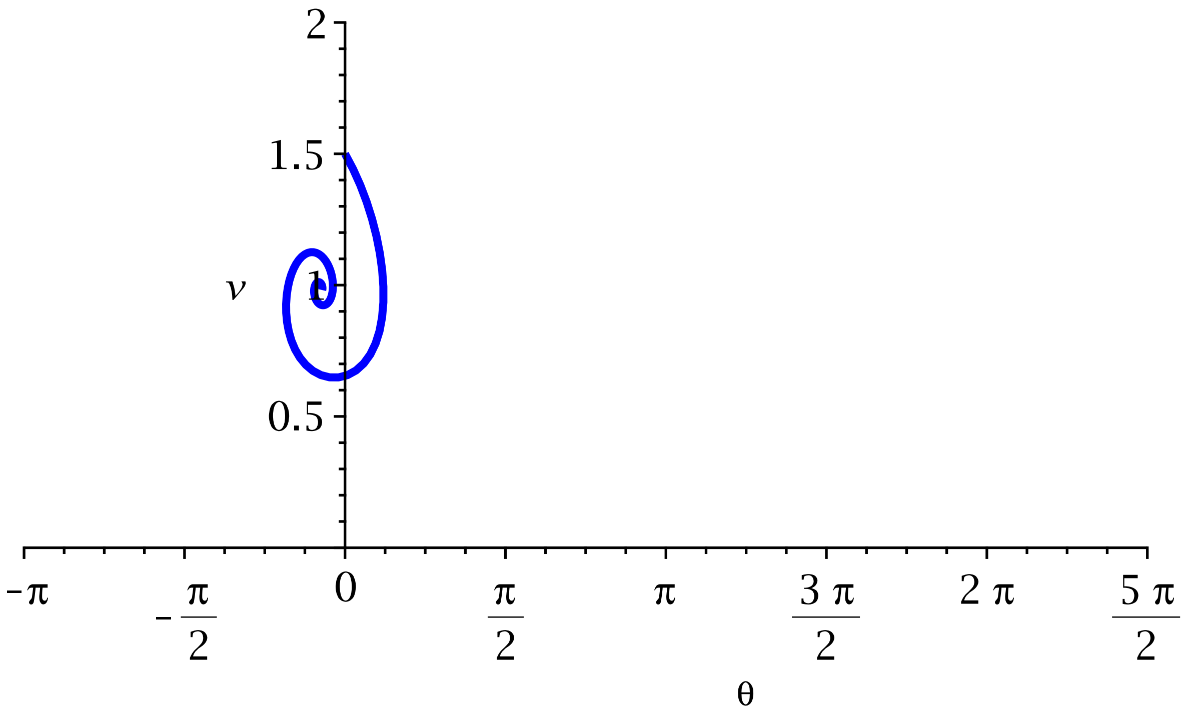
DEplot can no longer plot a direction field for us (since it lives in $\mathbb{R}^4$), but it can still plot the solutions in various slices, using scene. Note that a solution can cross itself in (x,y), since knowing x(0) and y(0) alone is not enough to know how the system evolves.

> $DEplot\left(xphug, [\theta(t), v(t), x(t), y(t)], t = 0..10,\right.$

$[[v(0) = 1.5, \theta(0) = 0, x(0) = 0, y(0) = 5]],$

$\theta = -\mathrm{Pi}..\frac{5\,\mathrm{Pi}}{2}, v = 0..2, x = -2..2, y = 0..8, tickmarks = [piticks, default],$

$\left.linecolor = blue, stepsize = .1, scene = [\theta, v]\right)$



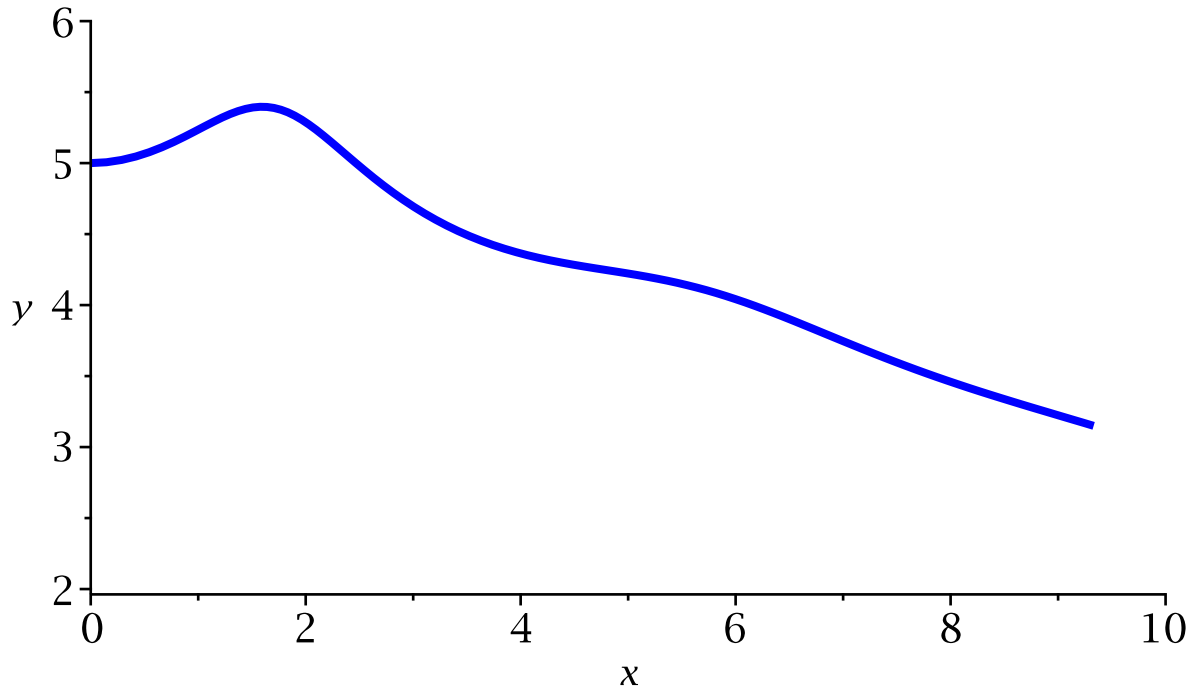
Here's how the above looks to an observer standing on the ground.

> $DEplot\left(xphug, [\theta(t), v(t), x(t), y(t)], t = 0..10,\right.$

$[[v(0) = 1.5, \theta(0) = 0, x(0) = 0, y(0) = 5]],$

$\theta = -\mathrm{Pi}..\frac{5\,\mathrm{Pi}}{2}, v = 0..2.5, x = 0..10, y = 2..6,$

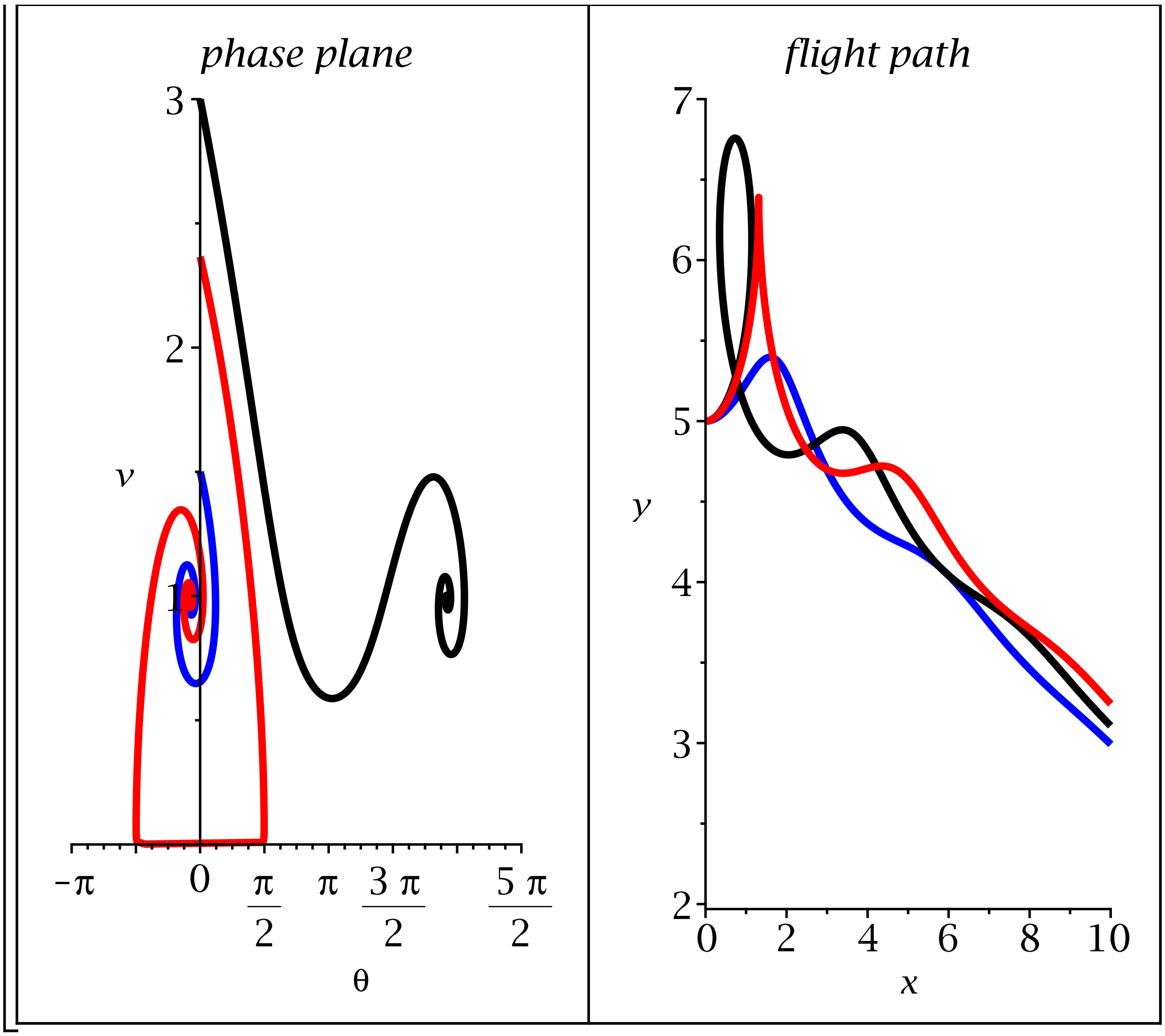
$\left.linecolor = blue, stepsize = .1, scene = [x, y]\right)$



Let's compare several initial conditions. After some trial and error, we see that v= 2.366 comes close to a stall.

$> \ inits := [[v(0) = 1.5, \theta(0) = 0, x(0) = 0, y(0) = 5], [v(0) = 3, \theta(0) = 0, x(0) = 0, y(0) = 5], [v(0) = 2.366, \theta(0) = 0, x(0) = 0, y(0) = 5]]:$

$display\left(\left\langle DEplot\left(xphug, [\theta(t), v(t), x(t), y(t)], t = 0..15, \ inits, \theta = -\mathrm{Pi}..\frac{5\,\mathrm{Pi}}{2}, v = 0..3, x = 0..10, y = 2..7, \ tickmarks = [piticks, default], linecolor = [blue, black, red], stepsize = .01, scene = [\theta, v], title = 'phase\ plane'\right) \middle| DEplot\left(xphug, [\theta(t), v(t), x(t), y(t)], t = 0..15, \ inits, \ \theta = -\mathrm{Pi}..\frac{5\,\mathrm{Pi}}{2}, v = 0..3, x = 0..10, y = 2..7, linecolor = [blue, black, red], stepsize = .01, scene = [x, y], title = 'flight\ path'\right)\right\rangle\right)$



phase plane
v
θ
3
2
1
−π
0
π/2
π
3π/2
5π/2
flight path
y
x
7
6
5
4
3
2
0
2
4
6
8
10