10/03/2019

Gliders and stuff.

Talk at the board. Too bad for you cuz you didn't come to class. Probably you should at least read the start of the text about it.

Some yammering:
As you know, sometimes we can do integrals and get a nice answer, like

> $\int x^2 \, \mathrm{d}x$

$$\frac{x^3}{3} \tag{1}$$

and sometimes the answers don't have an answer in terms of elementary functions, such as

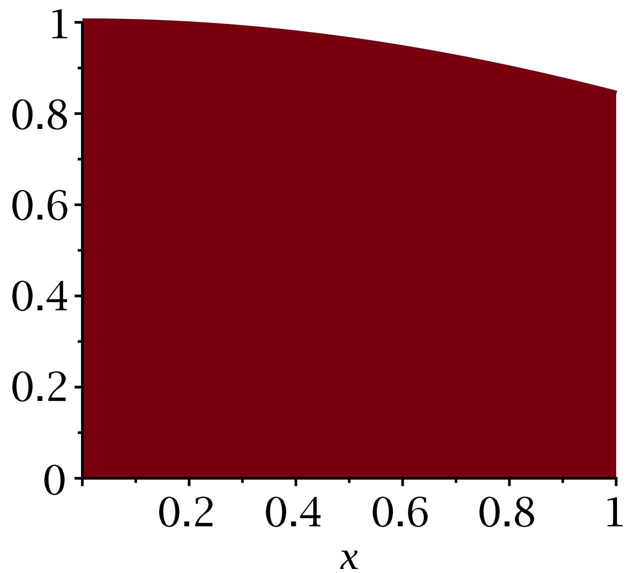
> $\int \frac{\sin(x)}{x} \, \mathrm{d}x$

$$\mathrm{Si}(x) \tag{2}$$

Above, Si(x) represents the "sine integral", which just means the function that represents $\int_0^x \frac{\sin(t)}{t} \, \mathrm{d}t$, that is, the integral is just its own self.

Of course, we can UNDERSTAND this integral, for example, by thinking of it as an area under a curve.

> $plot\left(\frac{\sin(x)}{x}, x = 0..1, filled = true, thickness = 3\right)$



Similarly, a differential equation might have a nice closed form solution, or not. Note that differential equations are just ways to solve integrals... that is, solving the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}(g(x)) = x^2$$

is the same thing as saying "find the function of x whose derivative is x^2", that is, find $\int x^2\, dx$

Of course, maple can do this just fine. Note that we can write g'(x) as `diff(g(x), x)`

```
> dsolve( diff(g(x), x) = x^2)
```

$$g(x) = \frac{x^3}{3} + \_C1 \tag{3}$$

If we have an initial condition, eg,

$$\left\{ \frac{d}{dx}(g(x)) = x^2 \ ,\ \ g(0) = 7 \right\}$$

we can have maple solve that too.

```
> dsolve({ diff(g(x), x) = x^2, g(0) = 7})
```

$$g(x) = \frac{x^3}{3} + 7 \tag{4}$$

Even if there are a few coupled equations, we might get lucky and there will be a nice solution:

```
> dsolve( {   diff(x(t), x) + y(t) = t·x(t),
              diff(y(t), y)  + (y(t))^3 = sin(t) })
```

$$\left[\left\{y(t) = \sin(t)^{1/3}, y(t) = -\frac{\sin(t)^{1/3}}{2} - \frac{I\sqrt{3}\,\sin(t)^{1/3}}{2}, y(t) = -\frac{\sin(t)^{1/3}}{2} + \frac{I\sqrt{3}\,\sin(t)^{1/3}}{2}\right\}, \left\{x(t) = \frac{y(t)}{t}\right\}\right] \tag{5}$$

But this won't always work.

Now let's go back to talking about gliders. See the start of the text to see where these equations come from. But they describe how the velocity and angle of a glider affected by gravity, lift, and drag behave.
R is a "coefficient of friction".

```
> phug := [ diff(v(t), t) = -sin(theta(t)) - R·v(t)^2,  diff(theta(t), t)
         = (v(t)^2-cos(theta(t)))/v(t) ]
```

$$phug := \left[\frac{d}{dt}\, v(t) = -\sin(\theta(t)) - R\, v(t)^2, \frac{d}{dt}\, \theta(t) = \frac{v(t)^2 - \cos(\theta(t))}{v(t)}\right] \tag{6}$$

We can ask for a solution, but I'm not happy with the "answer".

```
> dsolve(phug)
```

$$\left[\left\{\theta(t) = \_a \ \textbf{where} \left[\left\{\left(\frac{d}{d\_a}\, \_b(\_a)\right) \_b(\_a)\right.\right.\right.\right. \tag{7}$$

$$+\frac{1}{2\left(-\_b(\_a)+\sqrt{\_b(\_a)^2+4\cos(\_a)}\right)}\Big(\left(\_b(\_a)^2+4\cos(\_a)\right)^{3/2}R$$

$$+\_b(\_a)^2\sqrt{\_b(\_a)^2+4\cos(\_a)}\ R-2\ \_b(\_a)^3 R-8\ \_b(\_a)\cos(\_a)\ R$$

$$+4\sin(\_a)\sqrt{\_b(\_a)^2+4\cos(\_a)}+4\ \_b(\_a)\sin(\_a)\Big)=0\Big\},\ \Big\{\_a=\theta(t),$$

$$\_b(\_a)=\frac{d}{dt}\theta(t)\Big\},\ \Big\{t=\int\frac{1}{\_b(\_a)}\,d\_a+\_C1,\ \theta(t)=\_a\Big\}\Big]\Big],\ \Big\{v(t)=\frac{\frac{d}{dt}\theta(t)}{2}$$

$$-\frac{\sqrt{\left(\frac{d}{dt}\theta(t)\right)^2+4\cos(\theta(t))}}{2},\ v(t)=\frac{\frac{d}{dt}\theta(t)}{2}$$

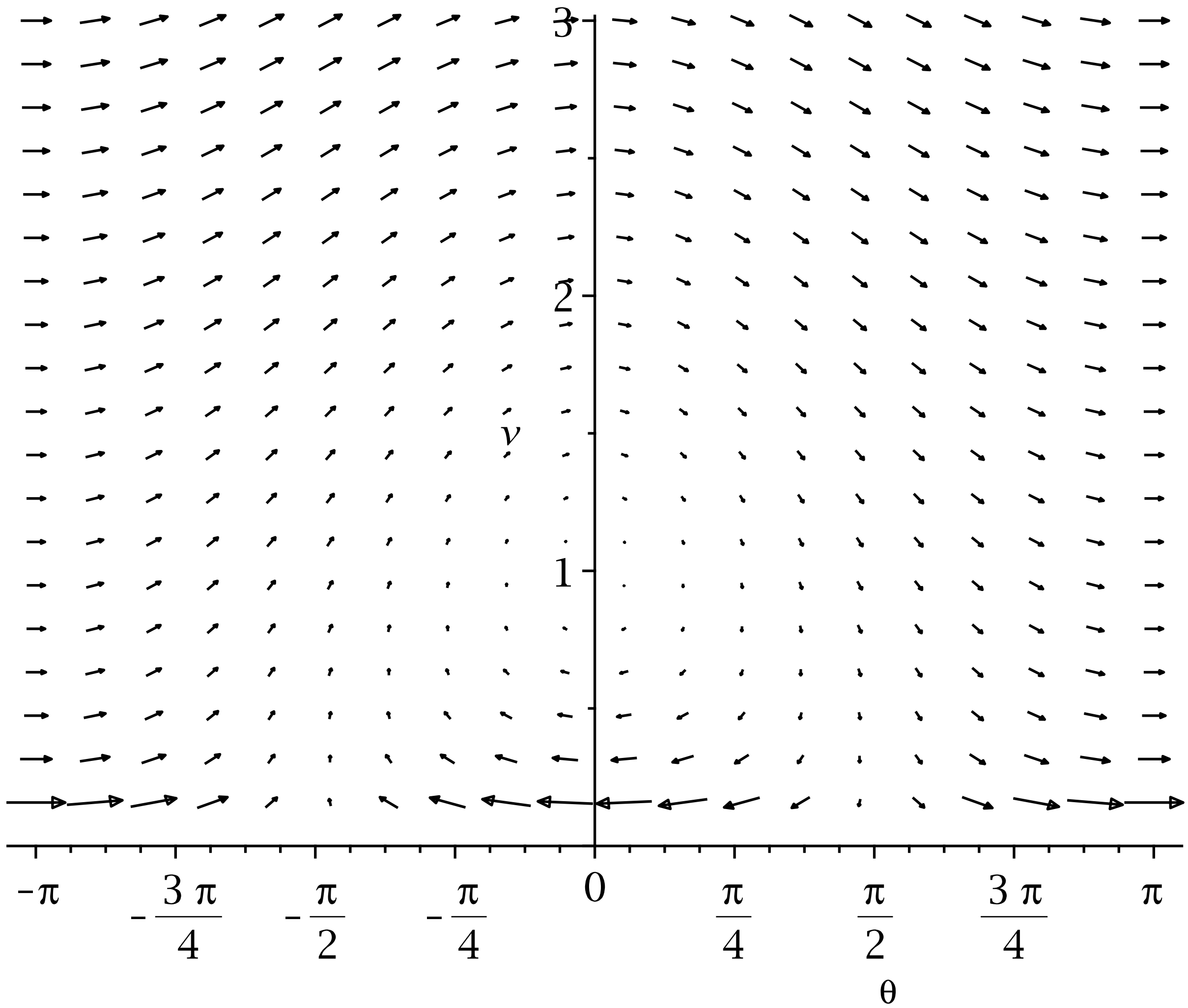
$$+\frac{\sqrt{\left(\frac{d}{dt}\theta(t)\right)^2+4\cos(\theta(t))}}{2}\Big\}\Big]\Big]$$

As I said, this is horrible, and doesn't actually tell us much since the solutions depend on derivatives, etc.... bleah!

Let's try something else. We can make a plot of the vector field, where at each point (theta,v), we put an arrow to indicate the direction the solution is going, and its length tells us how fast it is changing there.

Let's just do that for R=0, and we can ignore t.

```
> with(plots) :
> fieldplot([v - cos(theta)/v, -sin(theta)], theta = -Pi..Pi, v = 0..3,
      arrows = slim, tickmarks = [piticks, default])
```



Let's write this as a differential equation, which is a pain to type all the time.

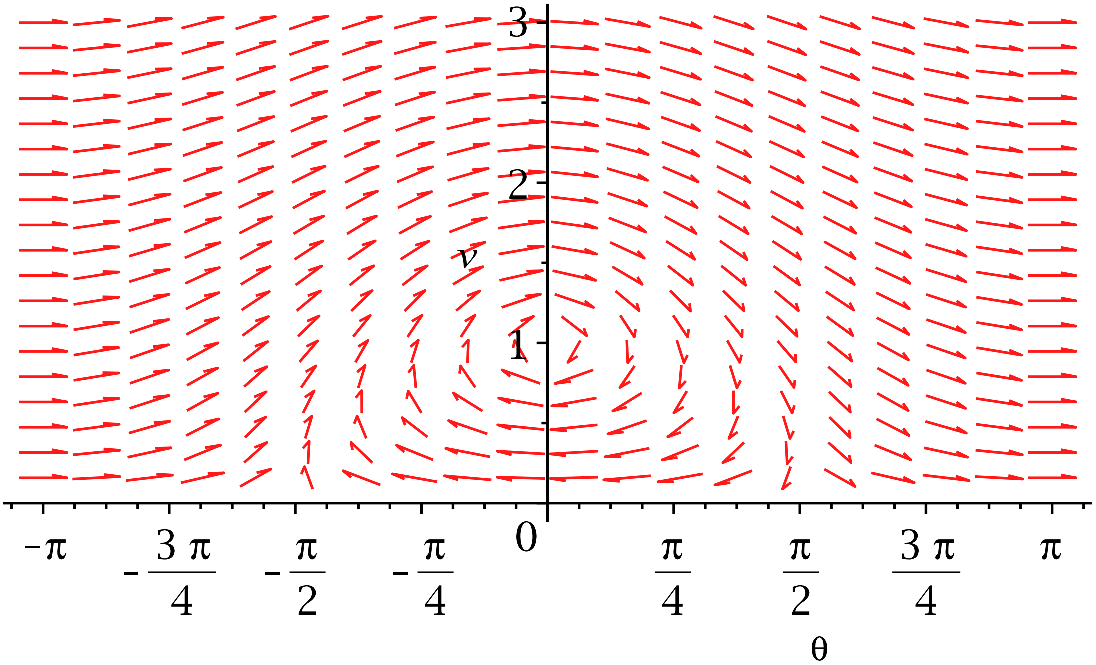
> $phug := \left[\, diff(v(t),\, t) = -\sin(\text{theta}(t)),\;\; diff(\text{theta}(t),\, t) = \dfrac{v(t)^2 - \cos(\text{theta}(t))}{v(t)} \,\right]$

$$phug := \left[\frac{\mathrm{d}}{\mathrm{d}t}\, v(t) = -\sin(\theta(t)),\; \frac{\mathrm{d}}{\mathrm{d}t}\, \theta(t) = \frac{v(t)^2 - \cos(\theta(t))}{v(t)}\right] \qquad \textbf{(8)}$$

Let's make a direction field-- a vector field where all the vectors have been rescaled to have length one. It is actually easier to see the solutions on the direction field.

> *with*(*DEtools*) :

> *DEplot*(*phug*,
[theta(*t*), *v*(*t*)], *t* = 0 ..1, theta = -Pi ..Pi, *v* = 0 ..3,
tickmarks = [*piticks*, *default*], *scaling* = *constrained*);



It is quite apparent, if you just look a little, that solutions near theta=0, v=1 "circle around it".
This corresponds to a regular up and down motion of the plane (theta increases and decreases) and an oscillating velocity.

For solutions with a higher velocity, theta is monotonically increasing, while the velocity rises and falls a little (so the plane is looping).
Let's ask maple to plot two of these: one starting at theta=0, v=1/2, and the other starting at
theta=−π (that is, flying the other way, upside-down) and v=2.

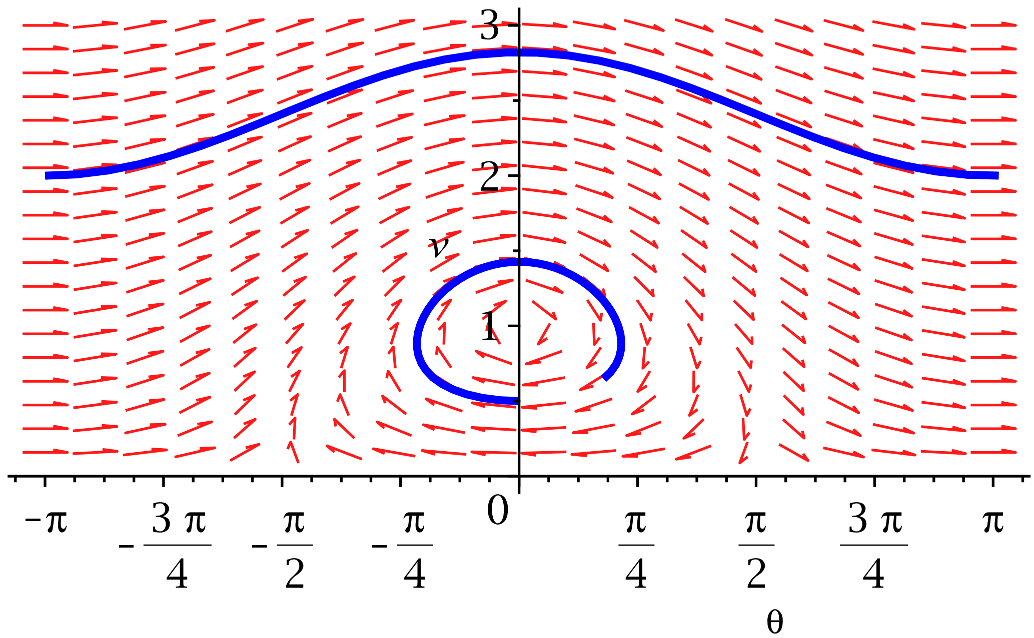
> $DEplot\Big(phug,$

$[\mathrm{theta}(t), v(t)],\ t = 0..4,$

$\Big[\Big[\mathrm{theta}(0) = 0, v(0) = \frac{1}{2}\Big], [\mathrm{theta}(0) = -\mathrm{Pi}, v(0) = 2]\Big],$

$\mathrm{theta} = -\mathrm{Pi}..\mathrm{Pi}, v = 0..3, tickmarks = [piticks, default], scaling = constrained,$

$linecolor = blue\Big);$



3
2
1
v
0
−π
- 3 π / 4
- π / 2
- π / 4
π / 4
π / 2
3 π / 4
π
θ

>