1 Section 4.1

Exercise 1 (#4). \( \lim_{x \to c} f(x) = L \iff \lim_{x \to 0} f(x + c) = L \)

\( \Rightarrow \): Suppose \( \lim_{x \to c} f(x) = L \). Write \( y = x - c \). If \( \epsilon > 0 \), then there exists \( \delta > 0 \) so that \( |x - c| < \delta \implies |f(x) - L| < \epsilon \). In terms of \( y \), this is \( |y| < \delta \implies |f(y + c) - L| < \epsilon \). Thus \( \lim_{y \to 0} f(y + c) = L \).

\( \Leftarrow \): Suppose \( \lim_{x \to 0} f(x + c) = L \). Write \( y = x + c \). If \( \epsilon > 0 \), then there exists \( \delta > 0 \) so that \( |x| < \delta \implies |f(x + c) - L| < \epsilon \). In terms of \( y \), this is \( |y - c| < \delta \implies |f(y) - L| < \epsilon \). Thus \( \lim_{y \to c} f(y + c) = L \).

Exercise 2 (#9).

(a). \( \lim_{x \to 1} \frac{1}{x} = -1 \)

We use the sequential criterion. Suppose \( x_n \to 2 \). Then \( \lim_{x \to 1} \frac{1}{x_n} = \frac{1}{1 - \lim_{x_n} x} = -1 \), by the limit laws.

(b). \( \lim_{x \to 1} \frac{x}{x + 1} = 1/2 \).

Again we use the sequential criterion and the limit laws. Suppose \( x_n \to 1 \). By applying the limit laws, we see that

\[
\lim_{x \to 1} \frac{x}{x + 1} = \frac{1}{1 + 1} = 1/2 \quad (1)
\]

(c). \( \lim_{|x|} \frac{x^2}{|x|} = 0 \)

Suppose \( \epsilon > 0 \). Then \( |x^2| = \frac{|x|^2}{|x|} = |x| \). So choose \( |x| < \delta \) and the result follows.

(d). \( \lim_{x \to 1} \frac{x^2 - x + 1}{x + 1} = 1/2 \)

We use the sequential criterion and the limit laws. Suppose \( x_n \to 1 \). Then

\[
\lim_{x \to 1} \frac{x^2_n - x_n + 1}{x_n + 1} = \frac{1 - 1 + 1}{1 + 1} = 1/2 \quad (2)
\]
Exercise 3 (#14). Let \( f(x) = x \) if \( x \in \mathbb{Q} \) and \( f(x) = 0 \) otherwise.

(a). \( \lim_{x \to 0} f(x) = 0. \)

We use the sequential criterion. Suppose \( x_n \to 0 \). Then \( |f(x_n)| \leq |x_n| \) since \( f(x_n) \) is either 0 or \( x_n \). Thus \( f(x_n) \to 0. \)

(b). If \( c \neq 0 \), then \( f(x) \) does not have a limit at \( c. \)

By the sequential criterion, we need to find a sequence \( x_n \to c \) so that \( f(x_n) \) does not converge. By theorem 2.4.8 (the density theorem) for any \( n \in \mathbb{N} \) we can choose a rational number \( y_n \) satisfying \( c - 1/n < y_n < c \). This gives a sequence of rational numbers \( y_n \) which converges to \( c. \) A similar argument applying corollary 2.4.9 gives a sequence of irrational numbers \( z_n \) converging to \( c. \) Now define \( x_n \) as follows: Put \( x_1 = y_1, x_2 = z_1, x_3 = y_2, \) etc... Then it follows that \( y_n \) and \( z_n \) are subsequences of \( x_n \). But \( f(y_n) \) clearly converges to 0 (check this) and \( f(z_n) \) clearly converges to \( c \) (check this as well). Since \( 0 \neq c \), this shows that \( f(x_n) \) has subsequences that converge to two different limits, and therefore cannot be a convergent sequence.

2 Section 4.2

Exercise 4 (#1).

All four of these are done in exactly the same way, so for brevity I will just do part a. The answers to all four parts are in the back of the book.

(a). \( \lim_{x \to 1} (x + 1)(2x + 3) = 10 \)

We have \( \lim_{x \to 1} (x + 1)(2x + 3) = \lim_{x \to 1} (x + 1) \lim_{x \to 1} (2x + 3) = (\lim_{x \to 1} x + 1)(\lim_{x \to 1} x + 3) = (1 + 1)(2 + 3) = 10. \)

Exercise 5 (#4). \( \lim_{x \to 0} \cos(1/x) \) does not exist but \( \lim_{x \to 0} x \cos(1/x) = 0. \)

By the sequential criterion, we need to find a sequence \( x_n \to 0 \) so that \( \cos(1/x_n) \) has no limit. Define \( x_n = \frac{1}{\pi n} \). Then \( \cos(1/x_n) = \cos(\pi n) = (-1)^n \), which does not converge. On the other hand \(-1 \leq \cos(1/x) < 1, \) so \(-x \leq x \cos(1/x) \leq x, \) so by the squeeze theorem and the sequential criterion \( \lim_{x \to 0} x \cos(1/x) = 0. \)

Exercise 6 (#9).

(a). If \( \lim_{x \to c} f \) and \( \lim_{x \to c} (f + g) \) exist, then \( \lim_{x \to c} g \) exists.

From the limit laws for functions, it follows that \( \lim_{x \to c} (f + g) - f \) exists. But \( (f + g) - f = g. \)

(b). If \( \lim_{x \to c} f \) and \( \lim_{x \to c} fg \) exist, does \( \lim_{x \to c} g \) exist?

No. Consider \( f(x) = 0 \) and \( g(x) = \frac{1}{x-c}. \)
Exercise 7 (#10). Give examples of functions $f$ and $g$ that do not have limits as $x \to c$, but $fg$ and $f+g$ do.

Consider $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = -1$ if $x \not\in \mathbb{Q}$, and consider $g(x) = -f(x)$. Then it is clear that $f$ and $g$ do not have limits as $x \to c$, for any $c$. On the other hand $(f+g)(x) = 0$ and $fg(x) = -f^2(x) = -1$. So the sum and product functions have limits for every $c$.

3 Section 4.3

Exercise 8 (#2). Give an example of a function with a right hand limit but no left hand limit.

Consider $f(x) = 1$ if $x \geq 0$ and $f(x) = -1$ if $x \in \mathbb{Q}$ and $x < 0$ and $f(x) = 1$ if $x \not\in \mathbb{Q}$ and $x < 0$.

Exercise 9 (#3). Put $f(x) = |x|^{-1/2}$ for $x \neq 0$. Then the right and left hand limits as $x \to 0$ are $+\infty$.

Notice that $f(x) = f(-x)$. Then it follows that the right and left hand limits, whatever they are, must be the same. (Check this from the definition). To show the limit is $+\infty$ we use the sequential criterion. Suppose $x_n \to 0$. Then it follows that $|x_n|^{1/2} \to 0$. So if $\epsilon > 0$, choose $N$ so that $n > N \implies |x_n|^{1/2} < \epsilon$. But then $|x_n|^{-1/2} > 1/\epsilon$. So for any $R \in \mathbb{R}$, we just have to choose $\epsilon$ so that $1/\epsilon > R$. Then there is $N \in \mathbb{N}$ so that $n \geq N \implies |x_n|^{-1/2} > R$.

Exercise 10 (#8). Suppose $f$ is defined for $x > 0$. Then $\lim_{x \to \infty} f(x) = L \iff \lim_{x \to 0^+} f(1/x) = L$.

$\Rightarrow$: If $\epsilon > 0$ there exists $R \in \mathbb{R}$ so that $x \geq R \implies |f(x) - L| < \epsilon$. Thus if $x < 1/R$ then $|f(1/x) - L| < \epsilon$.

$\Leftarrow$: If $\epsilon > 0$ there exists $\delta > 0$ so that $x < \delta \implies |f(1/x) - L| < \epsilon$. Write $y = 1/x$. Then if $y > \delta$ we have that $|f(y) - L| < \epsilon$. Thus $\lim_{y \to 0^+} f(y) = L$.