Solutions to Homework 8- MAT319

November 16, 2008

1 Section 4.1

Exercise 1 (#4). $\lim_{x\to c} f(x) = L \iff \lim_{x\to 0} f(x+c) = L$

$\Rightarrow$:

Suppose $\lim_{x\to c} f(x) = L$. Write $y = x - c$. If $\epsilon > 0$, then there exists $\delta > 0$ so that $|x-c| < \delta \implies |f(x) - L| < \epsilon$. In terms of y, this is $|y| < \delta \implies |f(y+c) - L| < \epsilon$. Thus $\lim_{y\to 0} f(y+c) = L$.

$\Leftarrow$:

Suppose $\lim_{x\to 0} f(x+c) = L$. Write $y = x + c$. If $\epsilon > 0$, then there exists $\delta > 0$ so that $|x| < \delta \implies |f(x+c) - L| < \epsilon$. In terms of y, this is $|y-c| < \delta \implies |f(y) - L| < \epsilon$. Thus $\lim_{y\to c} f(y) = L$.

Exercise 2 (#9).

(a). $\lim \frac{1}{1-x} = -1$

We use the sequential criterion. Suppose $x_n \to 2$. Then $\lim \frac{1}{1-x_n} = \frac{1}{1-\lim x_n} = -1$, by the limit laws.

(b). $\lim_{x\to 1} \frac{x}{x+1} = 1/2$.

Again we use the sequential criterion and the limit laws. Suppose $x_n \to 1$. By applying the limit laws, we see that

$$\lim_{x\to 1} \frac{x}{x+1} = \frac{1}{1+1} = 1/2 \tag{1}$$

(c). $\lim \frac{x^2}{|x|} = 0$.

Suppose $\epsilon > 0$. Then $|\frac{x^2}{|x|}| = \frac{|x|^2}{|x|} = |x|$. So choose $|x| < \delta$ and the result follows.

(d). $\lim_{x\to 1} \frac{x^2-x+1}{x+1} = 1/2$

We use the sequential criterion and the limit laws. Suppose $x_n \to 1$. Then

$$\lim \frac{x_n^2 - x_n + 1}{x_n + 1} = \frac{1-1+1}{1+1} = 1/2 \tag{2}$$

Exercise 3 (#14)**.** *Let $f(x) = x$ if $x \in \mathbb{Q}$ and $f(x) = 0$ otherwise.*

(a)**.** $\lim_{x\to 0} f(x) = 0$.

We use the sequential criterion. Suppose $x_n \to 0$. Then $|f(x_n)| \leq |x_n|$ since $f(x_n)$ is either 0 or x_n. Thus $f(x_n) \to 0$.

(b)**.** *If $c \neq 0$, then $f(x)$ does not have a limit at c.*

By the sequential criterion, we need to find a sequence $x_n \to c$ so that $f(x_n)$ does not converge. By theorem 2.4.8 (the density theorem) for any $n \in \mathbb{N}$ we can choose a rational number y_n satisfying $c - 1/n < y_n < c$. This gives a sequence of rational numbers y_n which converges to c. A similar argument applying corollary 2.4.9 gives a sequence of irrational numbers z_n converging to c. Now define x_n as follows: Put $x_1 = y_1$, $x_2 = z_1$, $x_3 = y_2$, etc... Then it follows that y_n and z_n are subsequences of x_n. But $f(y_n)$ clearly converges to 0 (check this) and $f(z_n)$ clearly converges to c (check this as well). Since $0 \neq c$, this shows that $f(x_n)$ has subsequences that converge to two different limits, and therefore cannot be a convergent sequence.

2 Section 4.2

Exercise 4 (#1)**.**

All four of these are done in exactly the same way, so for brevity I will just do part a. The answers to all four parts are in the back of the book.

(a)**.** $\lim_{x\to 1}(x+1)(2x+3) = 10$

We have $\lim_{x\to 1}(x+1)(2x+3) = \lim_{x\to 1}(x+1)\lim_{x\to 1}(2x+3) =$ $(\lim_{x\to 1} x + 1)(2\lim_{x\to 1} x + 3) = (1+1)(2+3) = 10$.

Exercise 5 (#4)**.** $\lim_{x\to 0}\cos(1/x)$ *does not exist but* $\lim_{x\to 0} x\cos(1/x) = 0$.

By the sequential criterion, we need to find a sequence $x_n \to 0$ so that $\cos(1/x_n)$ has no limit. Define $x_n = \frac{1}{\pi n}$. Then $\cos(1/x_n) = \cos(\pi n) = (-1)^n$, which does not converge. On the other hand $-1 \leq \cos(1/x) < 1$, so $-x \leq x\cos(1/x) \leq x$, so by the squeeze theorem and the sequential criterion $\lim_{x\to 0} x\cos(1/x) = 0$.

Exercise 6 (# 9)**.**

(a)**.** *If* $\lim_{x\to c} f$ *and* $\lim_{x\to c}(f+g)$ *exist, then* $\lim_{x\to c} g$ *exists.*

From the limit laws for functions, it follows that $\lim_{x\to c}(f+g) - f$ exists. But $(f+g) - f = g$.

(b)**.** *If* $\lim_{x\to c} f$ *and* $\lim_{x\to c} fg$ *exist, does* $\lim_{x\to c} g$ *exist?*

No. Consider $f(x) = 0$ and $g(x) = \frac{1}{x-c}$.

Exercise 7 (# 10)**.** *Give examples of functions f and g that do not have limits as $x \to c$, but fg and $f+g$ do.*

Consider $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = -1$ if $x \notin \mathbb{Q}$, and consider $g(x) = -f(x)$. Then it is clear that f and g do not have limits as $x \to c$, for any c. On the other hand $(f+g)(x) = 0$ and $fg(x) = -f^2(x) = -1$. So the sum and product functions have limits for every c.

3 Section 4.3

Exercise 8 (#2)**.** *Give an example of a function with a right hand limit but no left hand limit.*

Consider $f(x) = 1$ if $x \geq 0$ and $f(x) = -1$ if $x \in \mathbb{Q}$ and $x < 0$ and $f(x) = 1$ if $x \notin \mathbb{Q}$ and $x < 0$.

Exercise 9 (#3)**.** *Put $f(x) = |x|^{-1/2}$ for $x \neq 0$. Then the right and left hand limits as $x \to 0$ are $+\infty$.*

Notice that $f(x) = f(-x)$. Then it follows that the right and left hand limits, whatever they are, must be the same. (Check this from the definition). To show the limit is $+\infty$ we use the sequential criterion. Suppose $x_n \to 0$. Then it follows that $|x_n|^{1/2} \to 0$. So if $\epsilon > 0$, choose N so that $n > N \implies |x_n|^{1/2} < \epsilon$. But then $|x_n|^{-1/2} > 1/\epsilon$. So for any $R \in \mathbb{R}$, we just have to choose ϵ so that $1/\epsilon > R$. Then there is $N \in \mathbb{N}$ so that $n \geq N \implies |x_n|^{-1/2} > R$.

Exercise 10 (#8)**.** *Suppose f is defined for $x > 0$. Then $\lim_{x\to\infty} f(x) = L \iff \lim_{x\to 0^+} f(1/x) = L$.*

$\Rightarrow$: If $\epsilon > 0$ there exists $R \in \mathbb{R}$ so that $x \geq R \implies |f(x) - L| < \epsilon$. Thus if $x < 1/R$ then $|f(1/x) - L| < \epsilon$.
$\Leftarrow$: If $\epsilon > 0$ there exists $\delta > 0$ so that $x < \delta \implies |f(1/x) - L| < \epsilon$. Write $y = 1/x$. Then if $y > \delta$ we have that $|f(y) - L| < \epsilon$. Thus $\lim_{y\to 0+} f(y) = L$.