Solutions to Homework 7- MAT319

November 17, 2008

1 Section 3.5

Exercise 1 (#13). *Let $x_1 = 2$ and $x_{n+1} = 2 + 1/x_n$. Then x_n is contractive, and* $\lim x_n = 1 + \sqrt{2}$.

First let us show x_n is contractive. We have

$$|x_{n+2} - x_{n+1}| = |1/x_{n+1} - 1/x_n| = |\frac{x_n - x_{n+1}}{x_n x_{n+1}}| \leq \frac{1}{4}||x_{n+1} - x_n| \tag{1}$$

since it follows by induction that $x_n \geq 2$ (prove this for yourself). So it follows that x_n has a limit. We use the usual trick of noticing that the limit x must satisfy the equation $x = 2 + 1/x$. To make it easier to solve, we multiply both sides by x to arrive at $x^2 - 2x - 1 = 0$. By the quadratic formula the possible values for x are $1 \pm \sqrt{2}$. Since each term of the sequence is positive, we deduce that the limit must be $1 + \sqrt{2}$.

2 Section 3.6

Exercise 2 (# 7). *suppose x_n and y_n are sequences of positive terms with* $\lim \frac{x_n}{y_n} = 0$.

(a). *If $x_n \to +\infty$ then $y_n \to +\infty$.*

Suppose $R \in \mathbb{R}$. Since $\lim x_n = +\infty$, there is $N \in \mathbb{N}$ so that $n \geq N \implies |x_n| \geq R$. And thus $\frac{R}{y_n} \leq \frac{x_n}{y_n}$. Now choose $\epsilon < 1$. Then we may choose M so that $M > N$and so that if $n \geq M$ then $\frac{x_n}{y_n} < \epsilon = 1$. Combining these two inequalities we have $\frac{R}{y_n} < \frac{x_n}{y_n} < 1$. Thus $y_n > R$. Make sure you understand why the proof is completed.

(b). *Suppose y_n is bounded. Then* $\lim x_n = 0$.

Since $y_n > 0$, the fact that y_n is bounded means there is $R > 0$ so that $y_n < R$ for all n. Then for all n, we have $0 < \frac{x_n}{R} < \frac{x_n}{y_n}$. Now by the squeeze theorem it follows that $\lim \frac{x_n}{R} = 0$ and thus $\lim x_n = 0$.

Exercise 3 (#8).

(a).

We have $\sqrt{n^2+2} > n$, so it must be a divergent sequence.

(b).

$\frac{\sqrt{n}}{n^2+1} < \frac{n}{n^2+1} < \frac{1}{n}$, a convergent sequence.

(c).

The sequence in question is equal to $\sqrt{n+1/n}$ which is clearly greater than $\sqrt{n}$, a divergent sequence.

(d).

Consider the subsequence given by $n_k = n^2$. Thus $\sin n$ is a subsequence of $\sin\sqrt{n}$. By example 3.4.6c, this subsequence does not converge. Now by theorem 3.4.2 it follows that the original sequence $\sin\sqrt{n}$ does not converge.

3 Section 3.7

Exercise 4 (#4).

By hypothesis, the *sequences* $A_N = \sum_{n=1}^{N} x_n$ and $B_N = \sum_{n=1}^{N} y_n$ are convergent. By the limit law for sequences, we see that their sum $C_N = A_N + B_N$ is a convergent sequence as well. But $\lim_N C_N = \sum(x_n + y_n)$.

Exercise 5 (#5).

No. We have demonstrated in class that if a_n and $a_n + b_n$ are convergent sequences, then so is b_n. Thus, in the notation of the previous problem, if C_N is convergent, and if A_N is convergent, then so must be B_N. From the definition of A_N, B_N and C_N, this says that if $\sum(x_n + y_n)$ and $\sum x_n$ converge, then so must $\sum y_n$.

4 Section 4.1

Exercise 6 (#2).

(a). *Determine a condition on* $|x-4|$ *so that* $|\sqrt{x}-2| < 1/2$.

We have $|x-4| = |\sqrt{x}-2||\sqrt{x}+2|$, so that $|\sqrt{x}-2| = \frac{|x-4|}{|\sqrt{x}+2|} < \frac{|x-4|}{2}$, since $|\sqrt{x}+2| > 2$. So the condition is $|x-4| < 1$.

(b). *Determine a condition on* $|x-4|$ *so that* $|\sqrt{x}-2| < 1/100$.

From our work in a we see that the condition is $|x-4| < 1/50$.

Exercise 7 (#10).

(a).

Suppose $\epsilon > 0$. We wish to find δ so that $|x-2| < \delta \implies |x^2+4x-12| = |(x-2)(x+6)| = |(x-2)||(x-2)+8| \leq |(x-2)|(|x-2|+8) \leq \delta(\delta+8) < \epsilon$. Let us choose δ so that $\delta < \epsilon/9$, and so that $\delta < 1$. Then we have $\delta(\delta+8) < \delta(9) < (\epsilon/9)9 = \epsilon$.

(b).

Suppose $\epsilon > 0$. Notice that $|\frac{x+5}{2x+3} - 4| = |\frac{7(x+1)}{2(x+1)+1}|$. Now if $(x+1) > 0$, then $|\frac{7(x+1)}{2(x+1)+1}| < 7(x+1)$. So we just choose $|(x+1)| < \epsilon/7$. If $(x+1) < 0$, we choose $|(x+1)| < 1/4$ so that $|\frac{7(x+1)}{2(x+1)+1}| < \frac{7|(x+1)|}{2}$. So then we choose $|(x+1)| < 2\epsilon/7$.