Solutions to Homework 6- MAT319

November 10, 2008

1 Section 3.3

Exercise 1 (# 4). *Let* $x_1 = 1$ *and* $x_{n+1} = \sqrt{2 + x_n}$. *Then lim* $x_n = 2$

First we show that x_n is increasing by using an induction argument. $x_2 = \sqrt{3} > 1 = x_1$ so the base case holds. Now suppose that $x_n > x_{n-1}$. Then $x_n + 2 > x_{n-1} + 2$. Then $\sqrt{x_n + 2} > \sqrt{x_{n-1} + 2}$. Thus $x_{n+1} > x_n$. Now we show that x_n is bounded above by 2. We use induction again: $x_1 < 2$ so the base case holds. If $x_n < 2$, then $x_{n+1} = \sqrt{2 + x_n} < \sqrt{2 + 2} = 2$, as desired. Now by the monotone convergence theorem, x_n has a limit, say L. But then $L = \lim x_{n+1} = \lim \sqrt{x_n + 2} = \sqrt{(\lim x_n + 2)} = \sqrt{L + 2}$. So $L^2 - L - 2 = 0$. This equation has two solutions, namely $L = 2$, $L = -1$. Since $x_n > 0$ for all n, we deduce that $L = 2$.

Exercise 2 (# 7). *Suppose* $x_1 = a > 0$ *and* $x_{n+1} = x_n + 1/x_n$. *Then* x_n *diverges.*

We proceed by contradiction. Suppose that x_n converges. Then write $L = \lim x_n$. Then $L = \lim(x_n + 1/x_n) = \lim x_n + 1/\lim x_n$, so $L = L + 1/L$. Thus $0 = 1/L$. Since this is impossible, we deduce that x_n must diverge.

Exercise 3 (# 10). s_n *and* t_n *are monotone and if* $\lim s_n = \lim t_n$, *then* x_n *converges.*

First we show that $s_n = \sup\{x_k : x \geq n\}$ is decreasing. Notice that $s_n = \max\{x_n, s_{n+1}\}$. Thus $s_{n+1} \leq s_n$. Thus s_n is decreasing. Now we show that $t_n = \inf\{x_k : x \geq n\}$ is increasing. Notice that $t_n = \inf\{x_n, t_{n+1}\}$. Thus $t_n \leq t_{n+1}$. so t_n is a decreasing sequence. Now suppose that s_n and t_n converge. since $s_n \leq x_n \leq t_n$ and since $\lim s_n = \lim t_n$, by the squeeze theorem it follows that x_n converges.

Exercise 4 (# 15). *Calculate* $\sqrt{5}$, *correct to within* 5 *decimal places*

Following the example, set $s_{n+1} = 1/2(s_n + 5/s_n)$, and $s_1 = 5$. Then $s_2 = 3$, $s_3 = 7/3$, $s_4 = 2.23809524$, $s_5 = 2.2360689$. By the inequality $s_n - \sqrt{(5)} \leq (s_n^2 - 5)/s_n$ we see that

$$s_5 - \sqrt{5} \leq .0000018 \tag{1}$$

which tells us that s_5 is correct up to 5 decimal places.

2 Section 3.4

Exercise 5 (# 3). *let f_n be the fibonacci sequence and let $x_n = f_{n+1}/f_n$. Suppose that $L = \lim x_n$. What is the value of L?*

Recall that $f_{n+1} = f_n + f_{n-1}$ so that $x_n = f_{n-1}/f_{n+1} + 1$. Taking limits, we have $L = \lim f_{n-1}/f_{n+1} + 1 = 1/L + 1$. (Prove to yourself that $\lim f_{n-1}/f_n$ is the reciprocal of $\lim f_n/f_{n-1}$.) So $L = 1/L + 1$ and thus $L^2 - L - 1 = 0$. By the quadratic formula and the fact that $x_n > 0$ we deduce that $L = 1/2 + \sqrt{(5)}/2$.

Exercise 6 (#8).

(a). $\lim(3n)^{1/2n} = 1$

Notice that $(3n)^{1/2n} = (3/2)^{1/2n}(2n)^{1/2n}$. This is a subsequence (the even terms) of the sequence $(3/2)^{1/n}n^{1/n}$. Since $\lim n^{1/n} = 1$ and $\lim(3/2)^{1/n} = 1$ it follows that the limit of the product is the product of the limits, thus the limit is 1.

(b). $\lim(1 + 1/(2n))^{3n} = e^{3/2}$

Notice that $(1 + 1/(2n))^{3n} = (1 + 1/(2n))^{2n}(1 + 1/(2n))^n =$ $(1 + 1/(2n))^{2n}\sqrt{(1 + 1/(2n))^{2n})}$ But $(1 + 1/(2n))^{2n}$ is a subsequence (the even terms) of $(1 + 1/n)^n$ which converges to e. Thus

$$\lim(1 + 1/(2n))^{3n} = \lim\left((1 + 1/(2n))^{2n}\sqrt{(1 + 1/(2n))^{2n})}\right) = \tag{2}$$

$$\lim(1 + 1/(2n))^{2n} \lim\sqrt{(1 + 1/(2n))^{2n})} = e\sqrt{\lim(1 + 1/(2n)^{2n}} = \tag{3}$$

$$e\sqrt{e} = e^{3/2} \tag{4}$$

Exercise 7 (#9). *Suppose every subsequence of x_n has a subsequence that converges to 0. Then x_n converges to 0.*

By considering the contrapositive statement, it suffices to prove that if x_n does not converge to 0, then there exists a subsequence x_{n_k} that does not converge to 0. If x_n does not converge to 0, it follows from the definition that there exists $\epsilon > 0$ so that for all $N > 0$, there exists $n_k > N$ so that $|x_{n_k}| > \epsilon$. So we recursively construct a subsequence of x_n that does not converge to 0 as follows: choose $n_1 > 1$ so that $|x_{n_1}| > \epsilon$. Then to find $x_{n_{k+1}}$, choose $n_{k+1} > n_k$ so that $|x_{n_{k+1}}| > \epsilon$. (Here we are choosing $N = n_k$ in the definition above). Then it follows that $n_{k+1} \geq n_k$ so that x_{n_k} is actually a subsequence, and by construction, we see that x_{n_k} does not converge to 0.

3 Section 3.5

Exercise 8. *#2*

(a). *Show directly that $\frac{n+1}{n}$ is a Cauchy sequence.*

Suppose $\epsilon > 0$. Then choose N so that if $k > N$, $1/k < \epsilon/2$. Then notice that, for any $m, n > N$

$$|\frac{n+1}{n} - \frac{m+1}{m}| = |\frac{mn+m-nm-n}{mn} = |\frac{m-n}{mn}| \leq m/mn + n/mn = \tag{5}$$
$$1/n + 1/m < 2(\epsilon/2) = \epsilon. \tag{6}$$

so the sequence is cauchy.

(b). $1 + 1/2! + \ldots + 1/n!$ *is a Cauchy sequence.*

As an exercise for yourself, prove that $1/n! < 1/2^n$ as long as $n \geq 4$. Then, by using induction on m, prove that

$$1/2^n + 1/2^{n+1} + \ldots + 1/2^{n+m} \leq 1/2^{n-1} \tag{7}$$

Now if $\epsilon > 0$ is given, choose N so that if $n > N$ then $1/2^n < \epsilon$. Then if $n > m > N$, we have

$$|1 + 1/2! + \ldots + 1/n! - (1 + 1/2! + \ldots + 1/m!)| = \tag{8}$$
$$|1/(m+1)! + \ldots + 1/(m+n)!| \leq |1/2^{m+1} + \ldots + 1/2^{m+n}| \tag{9}$$
$$1/2^m \leq \epsilon \tag{10}$$

so the sequence is Cauchy.

Exercise 9 (#5). *Show that* $\lim |\sqrt{(n+1)} - \sqrt{n}| = 0$ *but* $\sqrt{n}$ *is not a Cauchy sequence.*

We have already shown that $|\sqrt{n+1} - \sqrt{n}|$ converges to 0 in a previous homework assignment. To show it is not Cauchy, choose $\epsilon = 1/4$, and suppose N is given. Choose $n = N$, and choose m so that $\frac{n}{2\sqrt{m}} < 1/4$ (we can do this since the sequence $1/2\sqrt{m}$ converges, and n is a fixed number.) Then

$$\sqrt{m} - \sqrt{n} = \frac{m-n}{\sqrt{m}+\sqrt{n}} \geq \frac{m-n}{2\sqrt{m}} = \frac{\sqrt{m}}{2} - \frac{n}{2\sqrt{m}} > \tag{11}$$
$$1/2 - \frac{n}{2\sqrt{m}} > 1/2 - 1/4 = 1/2 \tag{12}$$

So the sequence isn't cauchy.