Solutions to Homework 5- MAT319

October 26, 2008

1 3.1

Exercise 1 (3)**.**

This is just a straightforward calculation.

Exercise 2 (5)**.**

(a)**.** $\lim\left(\frac{n}{n^2+1}\right) = 0$

Notice that

$$\frac{n}{n^2+1} \leq \frac{n}{n^2} = \frac{1}{n}$$

So if $\epsilon > 0$ is given, then choose $N > 1/\epsilon$. Then if $n > N$, then

$$\frac{n}{n^2+1} \leq \frac{n}{n^2} = \frac{1}{n} < \epsilon$$

as desired.

(b)**.** $\lim\left(\frac{2n}{n+1}\right) = 2$

Suppose $\epsilon > 0$ is given. We wish to choose N so that if $n > N$, then we have

$$|\frac{2n}{n+1} - \frac{2(n+1)}{n+1}| = |\frac{-1}{n+1}| < \frac{1}{n} < \epsilon$$

It is clear that this inequality holds if we choose $N > 1/\epsilon$.

(c)**.**

If $\epsilon > 0$, we wish to find N so that if $n > N$ then

$$|\frac{3n+1}{2n+5} - 3/2| < |\frac{3n+1}{2n} - \frac{3n}{2n}| = \frac{1}{2n} < \epsilon$$

It is clear that the above inequality holds if we choose $N > \frac{1}{2\epsilon}$

(d).

If $\epsilon > 0$ is given, we wish to show that

$$\left|\frac{n^2-1}{2n^2+3} - 1/2\right| < \left|\frac{n^2-1}{2n^2} - \frac{n^2}{2n^2}\right| = 1/2n^2 < \epsilon$$

It is clear that the above inequality holds when we choose $N > \frac{1}{\sqrt{2\epsilon}}$.

Exercise 3 (11).

If $\epsilon > 0$, then we wish to choose N so that if $n > N$ then we have

$$\left|\frac{n+1-n}{n(n+1)}\right| = \frac{1}{n^2+n} < 1/n < \epsilon$$

So choose $N > 1/\epsilon$.

Exercise 4 (16). $\lim \frac{2^n}{n!} = 0$

First we prove the hint, that $2^n/n! \leq 2(2/3)^{n-2}$ if $n \geq 3$. For the base case, if $n = 3$ then we have $8/6 \leq 2(2/3)$. Suppose the result holds for n. Then

$$\frac{2(2^n)}{(n+1)n!} \leq \frac{2}{n+1} \cdot (2)(\frac{2}{3})^{n-2} \leq (2)(\frac{2}{3})^{n-1}$$

Now the result follows from example 3.1.11b.

2 3.2

Exercise 5 (6).

For a, Notice that $\lim(2+1/n) = 2 + \lim(1/n) = 2$, so that $\lim(2+1/n)^2 = \lim(2+1/n)\lim(2+1/n) = 4$. For b, we go back to the definition (just like in exercise 5 of the previous section) and choose $N > 1/\epsilon$. For c, rationalizing the numerator we find that

$$\frac{\sqrt{n}-1}{\sqrt{n}+1} = \frac{n-1}{n+2\sqrt{n}+1} = \frac{1-1/n}{1+2/\sqrt{n}+1/n} < \frac{1-1/n}{2/\sqrt{n}} < \frac{1}{1+2/\sqrt{n}}$$

Now this last term is a quotient of two convergent sequences, the constant sequence 1 and the sequence $1 + 2/\sqrt{n}$. Both of these sequences converge to 1, so their quotient converges to 1. For d, we have

$$\lim \frac{n+1}{n\sqrt{n}} = \lim \frac{1}{\sqrt{n}} + \lim \frac{1}{n\sqrt{n}} = 0$$

Exercise 6 (9). *y_n and $\sqrt{n}y_n$ converge, and find their limits.*

We have

$$y_n = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{\sqrt{n}}$$

Which converges to 0 by definition, with N chosen to be greater than $1/\epsilon^2$. On the other hand

$$\sqrt{n}y_n = \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{1+1/n}+1}$$

We know that $1+1/n$ converges to 1. By theorem 3.2.10, $\sqrt{1+1/n}$ converges to 1. So a quick application of the limit laws tells us that $\sqrt{n}y_n$ converges to 1/2.

Exercise 7 (13)**.**

For a, we have

$$1 \leq n^{1/n} \leq n$$

so that

$$1 \leq n^{1/n^2} \leq n^{1/n}$$

By 3.1.11d, $n^{1/n}$ converges to 1, so by the squeeze theorem the limit in question converges to 1. For b, notice that

$$1 \leq (n!)^{1/n^2} \leq (n^n)^{1/n^2} = n^{1/n}$$

So by the squeeze theorem, the limit is 1.

Exercise 8 (20)**.**

The hypothesis just tells us that x_n and $x_n - y_n$ are convergent sequences. By the addition limit law, we see that $x_n - (x_n - y_n) = y_n$ converges as well.