Solutions to Homework 5- MAT319

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1 3.1

Exercise 1 (3).
This is just a straightforward calculation.

Exercise 2 (5).
(a) \( \lim \left( \frac{n}{n^2+1} \right) = 0 \)
Notice that
\[
\frac{n}{n^2+1} \leq \frac{n}{n^2} = \frac{1}{n}
\]
So if \( \epsilon > 0 \) is given, then choose \( N > 1/\epsilon \). Then if \( n > N \), then
\[
\frac{n}{n^2+1} \leq \frac{n}{n^2} = \frac{1}{n} < \epsilon
\]
as desired.

(b) \( \lim \left( \frac{2n}{n+1} \right) = 2 \)
Suppose \( \epsilon > 0 \) is given. We wish to choose \( N \) so that if \( n > N \), then we have
\[
\left| \frac{2n}{n+1} - \frac{2(n+1)}{n+1} \right| = \left| \frac{-1}{n+1} \right| < \frac{1}{n} < \epsilon
\]
It is clear that this inequality holds if we choose \( N > 1/\epsilon \).

(c) If \( \epsilon > 0 \), we wish to find \( N \) so that if \( n > N \) then
\[
\left| \frac{3n+1}{2n+5} - 3/2 \right| < \left| \frac{3n+1}{2n} - \frac{3n}{2n} \right| = \frac{1}{2n} \leq \epsilon
\]
It is clear that the above inequality holds if we choose \( N > \frac{1}{2\epsilon} \).
(d).

If $\epsilon > 0$ is given, we wish to show that

$$\left| \frac{n^2 - 1}{2n^2 + 3} - \frac{1}{2} \right| < \left| \frac{n^2 - 1}{2n^2} - \frac{n^2}{2n^2} \right| = \frac{1}{2}n^2 < \epsilon$$

It is clear that the above inequality holds when we choose $N > \frac{1}{\sqrt{2\epsilon}}$.

**Exercise 3 (11).**

If $\epsilon > 0$, then we wish to choose $N$ so that if $n > N$ then we have

$$\left| \frac{n + 1 - n}{n(n + 1)} \right| = \frac{1}{n^2 + n} < \frac{1}{n} < \epsilon$$

So choose $N > 1/\epsilon$.

**Exercise 4 (16).** $\lim \frac{2^n}{n!} = 0$

First we prove the hint, that $2^n/n! \leq 2(2/3)^{n-2}$ if $n \geq 3$. For the base case, if $n = 3$ then we have $8/6 \leq 2(2/3)$. Suppose the result holds for $n$. Then

$$\frac{2(2^n)}{(n+1)n!} \leq \frac{2}{n+1} \cdot (2)(\frac{2}{3})^{n-2} \leq (2)(\frac{2}{3})^{n-1}$$

Now the result follows from example 3.1.11b.

### 2 3.2

**Exercise 5 (6).**

For $a$, Notice that $\lim(2 + 1/n) = 2 + \lim(1/n) = 2$, so that $\lim(2 + 1/n)^2 = \lim(2 + 1/n) \lim(2 + 1/n) = 4$. For $b$, we go back to the definition (just like in exercise 5 of the previous section) and choose $N > 1/\epsilon$. For $c$, rationalizing the numerator we find that

$$\sqrt{n} - 1 = \frac{n - 1}{n + 2\sqrt{n} + 1} = \frac{1 - 1/n}{1 + 2/\sqrt{n} + 1/n} < \frac{1 - 1/n}{2/\sqrt{n}} < \frac{1}{1 + 2/\sqrt{n}}$$

Now this last term is a quotient of two convergent sequences, the constant sequence 1 and the sequence $1 + 2/\sqrt{n}$. Both of these sequences converge to 1, so their quotient converges to 1. For $d$, we have

$$\lim \frac{n + 1}{n\sqrt{n}} = \lim \frac{1}{\sqrt{n}} + \lim \frac{1}{n\sqrt{n}} = 0$$

**Exercise 6 (9).** $y_n$ and $\sqrt{n}y_n$ converge, and find their limits.
We have
\[ y_n = \frac{1}{\sqrt{n + 1} + \sqrt{n}} < \frac{1}{\sqrt{n}} \]
Which converges to 0 by definition, with \( N \) chosen to be greater than \( 1/\epsilon^2 \). On the other hand
\[ \sqrt{n}y_n = \frac{\sqrt{n}}{\sqrt{n + 1} + \sqrt{n}} = \frac{1}{\sqrt{1 + 1/n + 1}} \]
We know that \( 1 + 1/n \) converges to 1. By theorem 3.2.10, \( \sqrt{1 + 1/n} \) converges to 1. So a quick application of the limit laws tells us that \( \sqrt{n}y_n \) converges to \( 1/2 \).

**Exercise 7 (13).**

For \( a \), we have
\[ 1 \leq n^{1/n} \leq n \]
so that
\[ 1 \leq n^{1/n^2} \leq n^{1/n} \]
By 3.1.11d, \( n^{1/n} \) converges to 1, so by the squeeze theorem the limit in question converges to 1. For \( b \), notice that
\[ 1 \leq (n!)^{1/n^2} \leq (n^n)^{1/n^2} = n^{1/n} \]
So by the squeeze theorem, the limit is 1.

**Exercise 8 (20).**

The hypothesis just tells us that \( x_n \) and \( x_n - y_n \) are convergent sequences. By the addition limit law, we see that \( x_n - (x_n - y_n) = y_n \) converges as well.