HW # 4

$\boxed{2.4}$ 18) $u>0$, $x<y$.

then $\frac{x}{u} < y/u$ by the Density theorem

there exists a rational number r, s.t. $x/u < r < y/u$

So $x < ru < y$.

$\boxed{2.5}$ 2) $S \subseteq \mathbb{R}$ is nonempty and bounded.

- $\Rightarrow$ By definition of the Bounded subsets of $\mathbb{R}$, S is bounded from above and below. let a and b be lower and upper bounds of S. respectively, we claim that $S \subseteq [a,b]$

$x \in S$. $x \leq b$, b is an upper bound $\rightarrow x \in [a,b]$
$x \geq a$ a is a lower bound

- $\Leftarrow$ if $S \subseteq I = [a,b]$ then S is bounded from above by b and bounded from below by a.

$x \in S \rightarrow x \in [a,b] \rightarrow$ $x \leq b \rightarrow x$ arbitrary element of S so b is an upper bound
$x \geq a \rightarrow x$ arbitrary element in S then a is a lower bound.

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6) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$, $I_n = [a_n, b_n]$

then $a_1 \leq a_2 \leq a_3 \leq \cdots$, $b_1 \geq b_2 \geq b_3 \geq \cdots$

let $P(n)$ be the statement $\{a_n \leq a_{n+1}$ and $b_n \geq b_{n+1}\}$

we would like to show $p(n)$ is true for every $n \in \mathbb{N}$

i) $n=1$, $a_1 \leq a_2$, $b_1 \geq b_2$

by assumption $[a_2, b_2] \subseteq [a_1, b_1]$

$a_2 \in [a_2, b_2] \Rightarrow a_2 \in [a_1, b_1] \Rightarrow$ $a_2 \geq a_1$

$b_2 \in [a_2, b_2] \Rightarrow b_2 \in [a_1, b_1] \Rightarrow$ $b_2 \leq b_1$

ii) if $p(n) \Rightarrow p(n+1)$, $I_{n+2} \subseteq I_{n+1}$

$[a_{n+2}, b_{n+2}] \subseteq [a_{n+1}, b_{n+1}]$

$a_{n+2} \in [a_{n+2}, b_{n+2}] \Rightarrow a_{n+2} \in [a_{n+1}, b_{n+1}] \Rightarrow a_{n+2} \geq a_{n+1}$

similarly $b_{n+2} \leq b_{n+1}$

Remark note that the statement of the problem can be rewritten as

$\forall n \in \mathbb{N}$, $a_n \leq a_{n+1}$, $b_n \geq b_{n+1}$

7).

$I_n = [0, 1/n], n \in \mathbb{N}, \quad \bigcap_{n=1}^{\infty} I_n = \{0\}$

i) $\{0\} \subseteq \bigcap_{n=1}^{\infty} I_n$;

for every $n \in \mathbb{N}$, $0 \in [0, 1/n] = I_n$, then $0 \in \bigcap_{n=1}^{\infty} I_n \Rightarrow \{0\} \subseteq \bigcap_{n=1}^{\infty} I_n$

ii) $\bigcap_{n=1}^{\infty} I_n \subseteq \{0\}$:

let a real number t belong to $\bigcap_{n=1}^{\infty} I_n$. we neet to show $t \in \{0\}$

or $t=0$. for the real number t, we have one of the following

1) $t < 0 \Rightarrow t \notin I_1 = [0,1] \Rightarrow t \notin \bigcap_{n=1}^{\infty} I_n$. Contradiction

2) $t > 0 \Rightarrow$ by Corollary 2.4.5, $\exists n \in \mathbb{N}$. $\frac{1}{n} < t$, so $t \notin [0, 1/n]$
therefor $t \notin \bigcap_{n=1}^{\infty} I_n$. Contradiction

3) $t=0$, then $t \in \{0\}$.

8) $J_n := (0, 1/n)$ for $n \in \mathbb{N}$, $\bigcap_{n=1}^{\infty} J_n = \phi$

let t be an element in $\bigcap_{n=1}^{\infty} J_n$, then $t \in (0, 1/n)$ for every $n \in \mathbb{N}$

i) $t < 0 \Rightarrow t \notin J_1 = (0,1)$, Contradiction

ii) $t = 0 \Rightarrow t \notin J_1 = (0,1)$. Conoradiction

iii) $t > 0 \Rightarrow \exists n$, $\frac{1}{n} < t \Rightarrow t \notin J_n = (0, \frac{1}{n})$, Contradiction.

So $\bigcap_{n=1}^{\infty} J_n$ Can not have any element.

12)

for $\frac{3}{8} \in [0,1]$

bisect $[0,1]$ $\rightarrow$ $[0, \frac{1}{2}]$ $\quad \frac{3}{8} \in [0, \frac{1}{2}] \Rightarrow a_1 = 0$

$\quad\quad\quad\quad\quad\;\;$ $\rightarrow$ $[\frac{1}{2}, 1]$

bisect $[0, \frac{1}{2}]$ $\rightarrow$ $[0, \frac{1}{4}]$

$\quad\quad\quad\quad\quad\;\;$ $\rightarrow$ $[\frac{1}{4}, \frac{1}{2}]$ $\quad \frac{3}{8} \in [\frac{1}{4}, \frac{1}{2}] \Rightarrow a_2 = 1$

$$\frac{\frac{1}{4} + \frac{1}{2}}{2} = \frac{\frac{3}{4}}{2} = \frac{3}{8}$$

bisect $[\frac{1}{4}, \frac{1}{2})$ $\rightarrow$ $[\frac{1}{4}, \frac{3}{8}]$ $\quad \frac{3}{8} \in [\frac{1}{4}, \frac{3}{8}] \Rightarrow a_3 = 0$

$\quad\quad\quad\quad\quad\;\;$ $\rightarrow$ $[\frac{3}{8}, \frac{1}{2}]$ $\quad \frac{3}{8} \in [\frac{3}{8}; \frac{1}{2}] \Rightarrow a_3 = 1$

two case $\rightarrow$ bisect $[\frac{1}{4}, \frac{3}{8}]$ $\rightarrow$ $[\frac{1}{4}, \frac{5}{16}]$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\rightarrow$ $[\frac{5}{16}, \frac{3}{8}]$ $\quad \frac{3}{8} \in [\frac{5}{16}, \frac{3}{8}] \Rightarrow a_4 = 1$

two case $\rightarrow$ bisect $[\frac{3}{8}, \frac{1}{2}]$ $\rightarrow$ $[\frac{3}{8}, \frac{7}{16}]$ $\quad \frac{3}{8} \in [\frac{3}{8}, \frac{7}{8}] \Rightarrow a_4 = 0$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\rightarrow$ $[\frac{7}{16}, \frac{1}{2}]$

now its clear that if we continue this process. if obtain

$$\frac{3}{8} = (.0101111\ldots)_2, \text{ or } \frac{3}{8} = (.0110000\cdots)_2, \quad \frac{3}{8} = \frac{1}{2^2} + \frac{1}{2^3}$$

$\frac{7}{16}$ is similar to the above one.

$a_k, b_k \in \{0, 1, \ldots, 9\}$

14) assume $\frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n} = \frac{b_1}{10} + \frac{b_2}{10^2} + \cdots + \frac{b_m}{10^m} \neq 0$

then $n = m$ and $a_k = b_k$

the problem should have said that $a_n \neq 0$, $b_m \neq 0$

otherwise the statement $n=m$ is not correct

for example $\frac{1}{10} + \frac{0}{10^2} = \frac{1}{10}$ & $2 \neq 1$

· let j be an integer bigger that n and m. then.

$$\frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n} + \frac{0}{10^{n+1}} + \cdots + \frac{0}{10^k} = \frac{b_1}{10} + \frac{b_2}{10^2} + \cdots + \frac{b_m}{10^m} + \frac{0}{10^{m+1}} + \cdots + \frac{0}{10^k}$$

then $\frac{a_1 - b_1}{10} + \frac{a_2 - b_2}{10^2} + \cdots + \quad = 0$

Consider the set of $k \in \mathbb{N}$ for which $a_k \neq b_k$, it has a least element denoted by j. that is $a_k = b_k$ for every $k < j$ and $a_j \neq b_j$. multiplying the last equality by 10^j one obtains

$$(a_j - b_j) + \frac{a_{j+1} - b_{j+1}}{10} + \frac{a_{j+2} - b_{j+2}}{10^2} + \cdots = 0,$$

a_i is assumed to be 0 for $i > m$ and $b_i = 0$ for $i > n$.

then $\frac{a_{j+1} - b_{j+1}}{10} + \frac{a_{j+2} - b_{j+2}}{10^2} + \cdots = b_j - a_j$

Now $b_j - a_j$ is an integer.

So $\dfrac{a_{j+1} - b_{j+1}}{10} + \dfrac{a_{j+2} - b_{j+2}}{10^2} + \cdots$ is an integer.

but $\left| \dfrac{a_{j+1} - b_{j+1}}{10} + \dfrac{a_{j+2} - b_{j+2}}{10^2} + \cdots \right|$

$\leq \left| \dfrac{a_{j+1} - b_{j+1}}{10} \right| + \left| \dfrac{a_{j+2} - b_{j+2}}{10^2} \right| + \cdots$

$\leq \dfrac{9}{10} + \dfrac{9}{10^2} + \cdots + \dfrac{0}{10^k} < 1$ (By induction)

So $\dfrac{a_{j+1} - b_{j+1}}{10} + \dfrac{a_{j+2} - b_{j+2}}{10^2} + \cdots + \dfrac{0}{10^k} = 0$

there for $b_j - a_j = 0 \Rightarrow b_j = a_j$ Contradiction.

17) $X = 1.25137137137\cdots$

$100X = 125.137137137\cdots$

$100000X = 125137.137137\cdots$

$\Rightarrow 100000X - 100X = 125137 - 125 = 125012$

then $X = \dfrac{125012}{100{,}000 - 100}$ rational.