HW # 4

2.4) 18) \( u > 0 , \ x < y \), then \( \frac{x}{u} < \frac{y}{u} \) by the Density Theorem.

there exists a rational number \( r \), s.t. \( \frac{r}{u} < \frac{y}{u} \)

so \( x < yr < y \).

2.5)

2) \( S \subseteq \mathbb{R} \) is non-empty and bounded.

- \( \Rightarrow \) By definition of the bounded subsets of \( \mathbb{R} \), \( S \) is bounded from above and below. Let \( a \) and \( b \) be lower and upper bounds of \( S \), respectively. We claim that \( S \subseteq [a,b] \)

\[ x \in S, \quad x \leq b, \quad b \text{ is an upper bound } \Rightarrow x \in [a,b] \]

\[ x \geq a, \quad a \text{ is a lower bound} \]

- \( \Leftarrow \) if \( S \subseteq [a,b] \), then \( S \) is bounded from above by \( b \) and bounded from below by \( a \).

\[ x \in S \Rightarrow x \in [a,b] \]

\[ x \geq a \quad \text{so } b \text{ is an upper bound} \]

\[ x \text{ arbitrary element of } S \]

\[ x \text{ arbitrary element in } S \]

\[ \text{then } a \text{ is a lower bound} \]
6) \( I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots, \quad I_n = [a_n, b_n] \)

then \( a \leq a_2 \leq a_3 \leq \cdots, \quad b_1 \geq b_2 \geq b_3 \geq \cdots \).

Let \( P(n) \) be the statement \( \{ a_n \leq a_{n+1} \text{ and } b_n \geq b_{n+1} \} \).

We would like to show \( P(n) \) is true for every \( n \in \mathbb{N} \).

i) \( n = 1 \), \( a_1 \leq a_2, \quad b_1 \geq b_2 \)

By assumption \([a_2, b_2] \subseteq [a_1, b_1] \).

\( a_2 \in [a_2, b_2] \Rightarrow a_2 \in [a_1, b_1] \Rightarrow a_2 \geq a_1 \).

\( b_2 \in [a_2, b_2] \Rightarrow b_2 \in [a_1, b_1] \Rightarrow b_2 \geq b_1 \).

ii) if \( P(n) \Rightarrow P(n+1) \), \( I_{n+2} \subseteq I_{n+1} \)

\([a_{n+2}, b_{n+2}] \subseteq [a_{n+1}, b_{n+1}] \).

\( a_{n+2} \in [a_{n+2}, b_{n+2}] \Rightarrow a_{n+2} \in [a_{n+1}, b_{n+1}] \Rightarrow a_{n+2} \geq a_{n+1} \).

Similarly \( b_{n+2} \leq b_{n+1} \).

Remark: Note that the statement of the problem can be rewritten as

\( \forall n \in \mathbb{N}, \quad a_n \leq a_{n+1}, \quad b_n \geq b_{n+1} \).
7) \( I_n = [0, \frac{1}{n}] \), new, \( \bigcap_{n=1}^{\infty} I_n = \{ 0 \} \)

i) \( \{ 0 \} \subseteq \bigcap_{n=1}^{\infty} I_n \):
for every new, \( 0 \in [0, \frac{1}{n}] = I_n \), then \( 0 \in \bigcap_{n=1}^{\infty} I_n = \{ 0 \} \subseteq \bigcap_{n=1}^{\infty} I_n \)

ii) \( \bigcap_{n=1}^{\infty} I_n \subseteq \{ 0 \} \):
let a real number \( t \) belong to \( \bigcap_{n=1}^{\infty} I_n \). We need to show \( t \in \{ 0 \} \)
or \( t = 0 \) for the real number \( t \), we have one of the following

1) \( t < 0 \Rightarrow t \notin I_1 = [0, 1] \Rightarrow t \notin \bigcap_{n=1}^{\infty} I_n \). *Contradiction*

2) \( t > 0 \Rightarrow \) by Corollary 2.4.5, for every \( \frac{1}{n} < t \), so \( t \notin [0, \frac{1}{n}] \) but for \( t \notin \bigcap_{n=1}^{\infty} I_n \). *Contradiction*

3) \( t = 0 \), then \( t \in \{ 0 \} \).

8) \( S_n := (0, \frac{1}{n}) \) for new, \( \bigcap_{n=1}^{\infty} S_n = \emptyset \)

let \( t \) be an element in \( \bigcap_{n=1}^{\infty} S_n \), then \( t \in (0, \frac{1}{n}) \) for every new

i) \( t < 0 \Rightarrow t \notin S_1 = (0, 1) \), *Contradiction*

ii) \( t = 0 \Rightarrow t \notin S_1 = (0, 1) \). *Contradiction*

iii) \( t > 0 \Rightarrow \exists n, \frac{1}{n} < t \Rightarrow t \notin S_n = (0, \frac{1}{n}) \), *Contradiction*.

So \( \bigcap_{n=1}^{\infty} S_n \) can not have any element.
12) for $\frac{3}{8} \in [0,1]
\begin{align*}
\text{bisect } [0,1] & \rightarrow [0, \frac{1}{2}] \quad \frac{3}{8} \in [0, \frac{1}{2}] \Rightarrow a_1 = 0 \\
\text{bisect } [0,\frac{1}{2}] & \rightarrow [\frac{1}{4}, \frac{1}{2}] \quad \frac{3}{8} \in [\frac{1}{4}, \frac{1}{2}] \Rightarrow a_2 = 1 \\
\text{bisect } [\frac{1}{4}, \frac{1}{2}] & \rightarrow [\frac{1}{8}, \frac{3}{8}] \quad \frac{3}{8} \in [\frac{1}{8}, \frac{3}{8}] \Rightarrow a_3 = 0
\end{align*}
\begin{align*}
\frac{1}{4} + \frac{1}{2} - \frac{2}{2} = \frac{2}{8} = \frac{1}{8}
\end{align*}
\begin{align*}
\text{bisect } [\frac{1}{8}, \frac{1}{2}] & \rightarrow [\frac{7}{16}, \frac{1}{2}] \quad \frac{3}{8} \in [\frac{7}{16}, \frac{1}{2}] \Rightarrow a_3 = 1
\end{align*}
\begin{align*}
\text{two case } & \text{ bisect } [\frac{1}{4}, \frac{3}{8}] \leftarrow [\frac{1}{4}, \frac{5}{16}] \quad \frac{1}{8} \in [\frac{5}{16}, \frac{3}{8}] \Rightarrow \alpha_4 = 1
\end{align*}
\begin{align*}
\text{bisect } [\frac{3}{8}, \frac{1}{2}] & \leftarrow [\frac{7}{16}, \frac{1}{2}] \quad \frac{1}{8} \in [\frac{7}{16}, \frac{1}{2}] \Rightarrow \alpha_4 = 0
\end{align*}

Now it's clear that if we continue this process, if obtain
\begin{align*}
\frac{3}{8} = (0.1011111...)_2, \text{ or } \frac{2}{8} = (0.0110000...)_2, \frac{3}{8} = \frac{1}{2} + \frac{1}{3}
\end{align*}

\frac{7}{16} \text{ is similar to the above one.}
\[ a_k, b_k \in \{0, 1, \ldots, 9\} \]

Assume
\[
\frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_n}{10^n} = \frac{b_1}{10} + \frac{b_2}{10^2} + \ldots + \frac{b_m}{10^m} + 0
\]

then \( n = m \) and \( a_k = b_k \)

the problem should have said that \( a_n + o, b_n + o \)

Otherwise the statement now is not correct.

For example \( \frac{1}{10} + \frac{0}{10^2} = \frac{1}{10} \neq 2 + 1 \)

Let \( j \) be an integer bigger than \( n \) and \( m \). Then
\[
\frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_n}{10^n} + 0 = \frac{b_1}{10} + \frac{b_2}{10^2} + \ldots + \frac{b_m}{10^m} + 0
\]

then \( \frac{a_1 - b_1}{10} + \frac{a_2 - b_2}{10^2} + \ldots + 0 = 0 \)

Consider the set of \( k \) for which \( a_k \neq b_k \). It has a least element denoted by \( j \). That is, \( a_k = b_k \) for every \( k < j \) and \( a_j \neq b_j \). Multiplying the last equality by \( 10^j \) one obtains
\[
(a_j - b_j) + \frac{a_{j+1} - b_{j+1}}{10} + \frac{a_{j+2} - b_{j+2}}{10^2} + \ldots + 0 = 0, \quad a_j \text{ is assumed to be } 0 \text{ for } i > m
\]

and \( b_i = 0 \) for \( i > n \).

Then
\[
\frac{a_{j+1} - b_{j+1}}{10} + \frac{a_{j+2} - b_{j+2}}{10^2} + \ldots = b_{j} - a_{j}
\]
Now $b_j - a_j$ is an integer.

So \[ \frac{a_{j+1} - b_{j+1}}{10} + \frac{a_{j+2} - b_{j+2}}{10^2} + \ldots \] is an integer.

but \[ \left| \frac{a_{j+1} - b_{j+1}}{10} \right| + \left| \frac{a_{j+2} - b_{j+2}}{10^2} \right| + \ldots \]

\[ \leq \frac{g}{10} + \frac{g}{10^2} + \ldots + \frac{0}{10^k} < 1 \quad \text{(By induction)} \]

So \[ \frac{a_{j+1} - b_{j+1}}{10} + \frac{a_{j+2} - b_{j+2}}{10^2} + \ldots + \frac{0}{10^k} = 0 \]

therefore $b_j - a_j > 0 \implies b_j = a_j$ contradiction.

17) $X = 1.25137137137 \ldots$

$100X = 125.137137137 \ldots$

$100000X = 125137.137137 \ldots$

$\implies 100000X - 100X = 125137 - 125 = 125012$

then $X = \frac{125012}{100000-100}$ rational.