Exersices for section 2.3

2) $S_2 = \{x \in \mathbb{R} : x > 0\}$.

- S_2 has lower bound, for example -1 is a lower bound for S_2.
- S_2 Does not have any upper bound, as if u is an upper bound for S_2 then u has to be bigger than 1 so it's positive. $u+1$ is positive so it is in S_2 but its not less than u, which is a contradiction.

- $\inf S_2 = 0$, i) for every x in S_2; $x > 0$ so 0 is a lower bound for S_2

  ii) 0 is the greatest lower bound. if u is another lower bound and $u > 0$ then $u/2 > 0$, there for $u/2 \in S_2$. that means u $(u > u/2)$ Can not be a lower bound.

- sup of S_2 Does not exist, because S_2 Does not have any upper bound by the first part.

4) $S_4 = \{1 - \frac{(-1)^n}{n} : n \in \mathbb{N}\} = \{2, \frac{1}{2}, \frac{4}{3}, \frac{3}{4}, \dots\}$

we claim that $\sup S_4 = 2$ and $\inf S_4 = \frac{1}{2}$

- $\sup S_4 = 2$. i) 2 is an upper bound for S_4

$$1 - \frac{(-1)^n}{h} \leq 2 \Leftrightarrow \frac{-(-1)^n}{n} \leq 1 \Leftrightarrow (-1)^{n+1} \leq n$$

but $(-1)^{n+1} \leq 1$ and $1 \leq n$ so $(-1)^{n+1} \leq n$

ii) 2 is the Least upper bound. if u is another upper bound then $u \geq 2$ since 2 is in S_4. there for 2 is the least upper bound.

6) $S \subseteq \mathbb{R}$, assume t is the upper bound of S contained in S.

we want to show that $t = \sup S$

i) t is an upper bound for S by assumption

ii) If r is another upper bound for S then r is bigger than every element in S, in particular, it has to be bigger than t since t is in S. therefore t is the least upper bound for S.

8) $S \subseteq \mathbb{R}$ is nonempty, $u = \sup S$

- for every $n \in \mathbb{N}$, $u - 1/n$. if $u - 1/n$ is an upper bound for S. as $u - 1/n < u$. u can not be the least upper bound for S.
- $u + 1/n$ is an upper bound for S. because if $x \in S$

$x \leq u$, u is an upper bound for S $\Rightarrow$ $x \leq u + 1/n$

$u \leq u + 1/n$

as x was arbitrary $u + 1/n$ is an upper bound for S.

11) $S \subseteq \mathbb{R}$, $s^* = \sup S$, $s^* \in S$, $u \notin S$.

then $\sup(S \cup \{u\}) = \sup\{s^*, u\}$

i) $\sup\{s^*, u\}$ is an upper bound for $S \cup \{u\}$,

because if $x \in S \cup \{u\}$ then $x \in \{u\} \Rightarrow x = u \Rightarrow x \leq u \Rightarrow x \leq \sup\{s^*, u\}$

or

$x \in S \Rightarrow x \leq s^* \Rightarrow x \leq \sup\{s^*, u\}$

11) Continued

ii) $\sup\{s^*, u\}$ is the least upper bound for $S \cup \{u\}$.

assume t is an upper bound for $S \cup \{u\}$ we need to show $\sup\{s^*, u\} \leq t$.

if t is an upper bound for $S \cup \{u\}$ then

$$u \leq t$$

and for every $x \in S$, $x \leq t$, since s^* is the least upper bound for S, t has to be bigger than s^*. As $\begin{matrix} u \leq t \\ s^* \leq t \end{matrix} \Rightarrow \sup\{s^*, u\} \leq t$

Remark: Note that we did not need the assumption s^* belongs to S and $u \notin S$. they are not necessary assumptions.

12) assume S is a non empty finite subset of $\mathbb{R}$. then $\sup S \in S$. we will show this by induction on the number of elements in S.

i) if S has only one element, say s^*. its easy to see that s^* is the sup of S (you have to do the details) so its in S.

ii) if the statement is true for n then it is true for $n+1$

let S be a subset of $\mathbb{R}$ with $n+1$ element. take an element x in S and define the set A to be $S \setminus \{x\}$ ($\setminus$ means minus)

Now A has n elements, so by assumption of induction its sup

12) ii) Continued.

belongs to A. by the previous problem

$\sup(S) = \sup(A \cup \{x\}) = \sup(\sup(A), x)$

Now if the sup of $\sup(A)$ and x is $\sup(A)$, then it belongs to A and since $A \subseteq S$ it belongs to S

if sup of $\sup(A)$ and x is x then its already in S.

$\boxed{2.4}$ 1) $\sup\{1 - 1/n ; n \in \mathbb{N}\} = 1$, $A := \{1 - 1/n ; n \in \mathbb{N}\}$

i) 1 is an upper bound for A, as $1 - 1/n < 1$ or $-1/n < 0$ or $\frac{1}{n} > 0$

ii) 1 is the least upper bound for A. if this is not true, there must be an upper bound, t, of A which is less than 1

$t \leq 1 \Rightarrow 1 - t \geq 0$ by Corollary 2.4.5 $\exists\, n \in \mathbb{N}$, $\frac{1}{n} \leq 1 - t$

hence $t \leq 1 - \frac{1}{n}$ which contradicts t being an upper bound for S.

3) let S be a non empty subset of $\mathbb{R}$ and u has the properties in the problem then $u = \sup S$

i) u is an upper bound for S:

take an arbitrary element x in S we need to prove $x \leq u$

by the second property, $u + 1/n$ is an upper bound for S then $x \leq u + 1/n$ for every n. then $x \leq u$ otherwise $\exists\, n \in \mathbb{N}$

3) Continued.

(by Corollary 2.4.5) $\frac{1}{n} < x-u \Rightarrow u+\frac{1}{n} < x$ which is a contradiction.

ii) u is the least upper bound for S:

if this is not true, there must be an upper bound t for S which is less than u,

$t<u$, $(2.4.5) \Rightarrow \exists n.\ 1/n < u-t \Rightarrow t < u-1/n$.

property on says that $u-1/n$ is not an upper bound for S.

So there must be an element y in S, s.t. $u-1/n<y$

Combining with $t< u-1/n$ we get $t<y$. So t can not be an upper bound for S.

6) $\sup(A+B) = \sup(A) + \sup(B)$

$\left.\begin{array}{l}\forall\ a\in A,\ a\le \sup(A) \\ \forall\ b\in B,\ b\le \sup(B)\end{array}\right\} \Rightarrow a+b \le \sup(A)+\sup(B)$

So $\sup(A)+\sup(B)$ is an upper bound for $A+B$, So, $\boxed{\sup(A+B) \le \sup(A)+\sup(B)}$

for the other direction. fix an arbitrary $b\in B$.

and observe. $\forall a\in A$, $a+b \le \sup(A+B)$, or $a \le \sup(A+B)-b$

then $\sup(A+B)-b$ is an upper bound for A, then.

$\sup(A) \le \sup(A+B)-b \Rightarrow b \le \sup(A+B) - \sup(A)$

now b is an arbitrary element in B so $\sup(A+B)-\sup(A)$ is an upper bound for B which implies $\sup(B) \le \sup(A+B) - \sup(A)$

or equivalently $\boxed{\sup(A)+\sup(B) \le \sup(A+B)}$.

6) the inf statement is similarly proved.

Remark One can first show that $\sup(-A) = -\inf(A)$, and then conclude the statement about inf's from the statement about sup's

7) $f: X \to \mathbb{R}$

$g: X \to \mathbb{R}$

$x \in X$ $\quad f(x) \leq \sup\{f(x) : x \in X\}$

$x \in X$ $\quad g(x) \leq \sup\{g(x) : x \in X\}$ $\Rightarrow$ $f(x) + g(x) \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$

so $\sup\{f(x)\} + \sup\{g(x)\}$ is an upper bound for $f(x)+g(x)$

hence $\sup\{f(x)+g(x)\} \leq \sup\{f(x)\} + \sup\{g(x)\}$.

inf statement is similar.

for examples let $X = [-2, 2]$.

Ex 1 $f(x) \equiv 2$, $g(x) \equiv 3$ $\Rightarrow$ equalities.

Ex 2. $f(x) = x$, $g(x) = -x$, both sup & inf gives inequalities.

Ex 3. $f(x) = \begin{cases} 1 & -2 \leq x \leq 0 \\ -x+1 & 0 < x \leq 2 \end{cases}$

$g(x) = \begin{cases} 2x+1 & -2 \leq x \leq 0 \\ 1 & 0 < x \leq 2 \end{cases}$

equality for sup's

inequality for inf's