1 Section 1.3

Exercise 1 (# 4). Exhibit a bijection between $\mathbb{N}$ and odd integers greater than 13.

The bijection is $f(n) = 2n + 13$. It is injective, for if $f(k_1) = f(k_2)$, then $2k_1 + 13 = 2k_2 + 13$, thus $k_1 = k_2$. It is surjective, for if $k$ is an odd integer greater than 13, then $k-12$ is an odd integer greater than 1, thus $k-12 = 2n+1$. But then $k = 2n + 1 + 12 = 2n + 13$ as desired.

Exercise 2 (# 11). If $|S| = n$ then $\mathcal{P}(S)$ has $2^n$ elements.

Base case ($n = 1$): Suppose $S = \{a\}$. Then $\mathcal{P}(S) = \{\emptyset, \{a\}\}$

Induction Step: Suppose $|S| = n \Rightarrow |\mathcal{P}(s)| = 2^n$, and suppose that $|J| = n + 1$. Let $a \in J$ be arbitrary. Then $\mathcal{P}(J)$ is the collection of subsets of $J$ which contain $a$ and the subsets of $J$ that don’t. By the induction hypothesis, there are precisely $2^n$ subsets of $J$ which contain $a$. Since the sets that don’t contain $a$ are precisely the complements of the ones that do, there are $2^n$ of those as well. Thus $|\mathcal{P}(s)| = 2^n + 2^n = 2^{n+1}$.

2 Section 2.1

Exercise 3 (# 3). Solve $2x + 5 = 8$ by using the field axioms of $\mathbb{R}$.

For the sake of brevity, we just do (a). The other equations are solved in a
similar manner.

\[ 8 = 2x + 5 \quad (1) \]
\[ 8 - 5 = 2x + 5 - 5 \text{ (see definition of subtraction) } \quad (2) \]
\[ 3 = 2x + (5 - 5) \text{ (associativity) } \quad (3) \]
\[ 3 = 2x + 0 \text{ (existence of negatives) } \quad (4) \]
\[ 3 = 2x \text{ (additive identity) } \quad (5) \]
\[ \frac{1}{2}3 = \frac{1}{2}2x \quad (6) \]
\[ \frac{3}{2} = (\frac{1}{2}2)x \text{ (associativity) } \quad (7) \]
\[ \frac{3}{2} = 1x \text{ (multiplicative inverse) } \quad (8) \]
\[ \frac{3}{2} = x \text{ (multiplicative identity) } \quad (9) \]

Exercise 4 (**# 4**). If \( a \in \mathbb{R} \) satisfies \( a \cdot a = 0 \) then \( a = 0 \) or \( a = 1 \).

First notice that \( a = 0 \) is a solution to the equation (see the axiom on existence of a 0 element). To find other solutions, suppose \( a \neq 0 \). Then \( \frac{1}{a} \) exists, and so \( \frac{1}{a} \cdot a = \frac{1}{a}a \). Using the field axioms it is easy to show that this implies that \( a = 1 \).

Exercise 5 (**#8**).

(a). If \( x, y \in \mathbb{Q} \) then \( x + y, xy \in \mathbb{Q} \).

Write \( x = a/b \) and \( y = c/d \). Then \( x + y = a/b + c/d = (ad + cb)/bd \in \mathbb{Q} \). Then \( xy = a/b \cdot c/d = ac/bd \in \mathbb{Q} \).

(b). If \( x \in \mathbb{Q} \) and \( y \notin \mathbb{Q} \), then \( x + y \notin \mathbb{Q} \). If \( x \neq 0 \) then \( xy \notin \mathbb{Q} \).

Write \( x = a/b \). For contrapositive, suppose \( x + y \in \mathbb{Q} \). Then write \( x + y = c/d \). But then \( y = c/d - a/b \in \mathbb{Q} \). If \( x \neq 0 \), then if \( xy \in \mathbb{Q} \), we have \( (a/b) \cdot y \in \mathbb{Q} \). Since \( a \neq 0 \), then (by part (a)) \( b/a \cdot (a/b)y \in \mathbb{Q} \). Thus \( y \in \mathbb{Q} \).

Exercise 6 (**#9**).

(a).

Write \( x_1 = s_1 + t_1 \sqrt{2} \) and \( x_2 = s_2 + t_2 \sqrt{2} \). Then \( x_1 + x_2 = (s_1 + s_2) + (t_1 + t_2) \sqrt{2} \in K \). Then \( x_1x_2 = (s_1s_2 + 2t_1t_2) + (2s_1t_1) \sqrt{2} \in K \).

(b).

If \( x \neq 0 \), write \( x = s + t\sqrt{2} \), where either \( s \neq 0 \) or \( t \neq 0 \). Then \( \frac{1}{s + t\sqrt{2}} = \frac{s - t\sqrt{2}}{s^2 - 2t^2} = \frac{s}{s^2 - 2t^2} + \frac{-t}{s^2 - 2t^2} \sqrt{2} \in K \).

Exercise 7 (**#18**). If \( a \leq b + \epsilon \) for all \( \epsilon > 0 \), then \( a \leq b \).
Suppose to the contrary that \( a > b \). Then choose \( \epsilon = \frac{a-b}{2} \). Then \( a \leq b + \epsilon = b + \frac{a-b}{2} \Rightarrow a - b \leq \frac{a-b}{2} \). This is a contradiction since \( a - b \geq 0 \) by hypothesis.

**Exercise 8** (#23). If \( a > 0 \), \( b > 0 \), then \( a < b \iff a^n < b^n \).

We proceed by induction. The base case is obvious. For the induction step, suppose that \( a < b \iff a^k < b^k \). We wish to prove that \( a < b \iff a^{k+1} < b^{k+1} \).

\[
\Rightarrow \quad \text{If } a < b, \text{ then by the induction hypothesis, } a^k < b^k. \text{ Then, since } a > 0, \text{ we have } a^{k+1} < ab^k. \text{ Then, by Thm 2.1.7c, since } a < b, \text{ we have } ab^k < b^{k+1}, \text{ establishing the result.}
\]

\[
\Leftarrow \quad \text{Now suppose } a^{k+1} < b^{k+1}, \text{ which is to say that } b^{k+1} - a^{k+1} > 0. \text{ Using the axioms of the real numbers one can show that } b^{k+1} - a^{k+1} = (b-a)(b^k + a^k). \text{ Since } b^{k+1} - a^{k+1} > 0 \text{ and since } b^k + a^k > 0 \text{ it follows from the order axioms that } b - a > 0, \text{ which is the desired result.}
\]

### 3 Section 2.2

**Exercise 9** (#2). \( |a+b| < |a| + |b| \iff ab > 0 \)

First notice that from the order axioms we can show that \( ab > 0 \iff a > 0 \text{ and } b > 0 \text{ or } a < 0 \text{ and } b < 0. \)

\[
\Leftarrow \quad \text{There are two cases: Either } a > 0, b > 0, \text{ or } a < 0, b < 0. \text{ In the first case we have } |a+b| = a + b = |a| + |b| \text{ as desired. In the second case } -|a+b| = a + b = -|a| - |b|, \text{ so multiplying both sides by } -1 \text{ gives the result.}
\]

\[
\Rightarrow \quad \text{We prove the contrapositive, which is an equivalent statement: If } a > 0, b < 0, \text{ then } |a+b| \neq |a| + |b|. \text{ By definition, either } |a+b| = a + b \text{ or } |a+b| = -(a+b). \text{ In the first case, the result follows as soon as we show that } a + b \neq |a| + |b| = a - b, \text{ which is equivalent to showing that } b \neq -b, \text{ which follows immediately since } b \neq 0. \text{ In the second case, we wish to show that } -(a + b) \neq |a| + |b| = a - b, \text{ which is equivalent to showing that } a \neq -a, \text{ which again follows since } a \neq 0.
\]

**Exercise 10** (#5). If \( a < x < b \text{ and } a < y < b \text{ then } |x - y| < b - a. \)

Geometrically, this just says that the distance from \( y \) to \( x \) is less than the distance from \( b \) to \( a \). Without loss of generality, let us assume that \( x < y \) (if this is not so, then we can just switch the letters \( x \) and \( y \)). Then we wish to show that \( y - x < b - a \). Since \( y < b \) and \( a < x \) it follows that \( y - x < b - a. \)

**Exercise 11** (#14). If \( \epsilon > 0 \text{ and } \delta > 0, \text{ then } \text{ (1) } V_\epsilon(a) \cap V_\delta(a) = V_{\min\{\epsilon, \delta\}}(a) \text{ and (2) } V_\epsilon(a) \cup V_\delta(a) = V_{\max\{\epsilon, \delta\}}(a) \text{ for appropriate choices of } \gamma.

Proof. For (1), \( \gamma = \min\{\epsilon, \delta\} \), for then it is clear that \( |x-a| < \gamma \iff |x-a| < \delta \text{ and } |x-a| < \epsilon. \) For (2), \( \gamma = \max\{\epsilon, \delta\} \), for then it is clear that \( |x-a| < \gamma \iff |x-a| < \delta \text{ or } |x-a| < \epsilon. \)