Solutions to Homework 2- MAT319/320

September 28, 2008

1 Section 1.3

Exercise 1 (# 4)**.** *Exhibit a bijection between* $\mathbb{N}$ *and odd integers greater than* 13.

The bijection is $f(n) = 2n + 13$. It is injective, for if $f(k_1) = f(k_2)$, then $2k_1 + 13 = 2k_2 + 13$, thus $k_1 = k_2$. It is surjective, for if k is an odd integer greater than 13, then $k-12$ is an odd integer greater than 1, thus $k-12 = 2n+1$. but then $k = 2n + 1 + 12 = 2n + 13$ as desired.

Exercise 2 (# 11)**.** *If* $|S| = n$ *then* $\mathcal{P}(S)$ *has* 2^n *elements.*

Base case (n = 1): Suppose $S = \{a\}$. Then $\mathcal{P}(S) = \{\emptyset, \{a\}\}$

Induction Step:
Suppose $|S| = n \Rightarrow |\mathcal{P}(s)| = 2^n$, and suppose that $|J| = n + 1$. Let $a \in J$ be arbitrary. Then $\mathcal{P}(J)$ is the collection of subsets of J which contain a and the subsets of J that don't. By the induction hypothesis, there are precisely 2^n subsets of J which contain a. Since the sets that don't contain a are precisely the complements of the ones that do, there are 2^n of those as well. Thus $|\mathcal{P}(s)| = 2^n + 2^n = 2^{n+1}$.

2 Section 2.1

Exercise 3 (# 3)**.** *Solve* $2x + 5 = 8$ *by using the field axioms of* $\mathbb{R}$.

For the sake of brevity, we just do (a). The other equations are solved in a similar manner.

$$
\begin{aligned}
8 &= 2x+5 && (1)\\
8-5 &= 2x+5-5 \text{ (see definition of subtraction)} && (2)\\
3 &= 2x+(5-5) \text{ (associativity)} && (3)\\
3 &= 2x+0 \text{ (existence of negatives)} && (4)\\
3 &= 2x \text{ (additive identity)} && (5)\\
\frac{1}{2}3 &= \frac{1}{2}2x && (6)\\
\frac{3}{2} &= (\frac{1}{2}2)x \text{ (associativity)} && (7)\\
\frac{3}{2} &= 1x \text{ (multiplicative inverse)} && (8)\\
\frac{3}{2} &= x \text{ (multiplicative identity)} && (9)
\end{aligned}
$$

Exercise 4 (# 4)**.** *If $a \in \mathbb{R}$ satisfies $a \cdot a = 0$ then $a = 0$ or $a = 1$.*

First notice that $a = 0$ is a solution to the equation (see the axiom on existence of a 0 element). To find other solutions, suppose $a \neq 0$. Then $\frac{1}{a}$ exists, and so $\frac{1}{a}a \cdot a = \frac{1}{a}a$. Using the field axioms it is easy to show that this implies that $a = 1$.

Exercise 5 (#8)**.**

(a)**.** *If $x, y \in \mathbb{Q}$ then $x + y, xy \in Q$.*

Write $x = a/b$ and $y = c/d$. Then $x + y = a/b + c/d = (ad + cb)/bd \in \mathbb{Q}$. Then $xy = a/b \cdot c/d = ac/bd \in \mathbb{Q}$.

(b)**.** *If $x \in \mathbb{Q}$ and $y \notin \mathbb{Q}$, then $x + y \notin \mathbb{Q}$. If $x \neq 0$ then $xy \notin \mathbb{Q}$.*

Write $x = a/b$. For contrapositive, suppose $x + y \in \mathbb{Q}$. Then write $x + y = c/d$. But then $y = c/d - a/b \in \mathbb{Q}$. If $x \neq 0$, then if $xy \in \mathbb{Q}$, we have $(a/b) \cdot y \in \mathbb{Q}$. Since $a \neq 0$, then (by part (a)) $b/a \cdot (a/b)y \in \mathbb{Q}$. Thus $y \in \mathbb{Q}$.

Exercise 6 (# 9)**.**

(a)**.**

Write $x_1 = s_1 + t_1\sqrt{2}$ and $x_2 = s_2 + t_2\sqrt{2}$. Then $x_1 + x_2 = (s_1 + s_2) + (t_1 + t_2)\sqrt{2} \in K$. Then $x_1x_2 = (s_1s_2 + 2t_1t_2) + (2s_1t_1)\sqrt{2} \in K$.

(b)**.**

If $x \neq 0$, write $x = s + t\sqrt{2}$, where either $s \neq 0$ or $t \neq 0$. Then $\frac{1}{s+t\sqrt{2}} = \frac{1}{s+t\sqrt{2}}\frac{s-t\sqrt{2}}{s-t\sqrt{2}} = \frac{s-t\sqrt{2}}{s^2-2t^2} = \frac{s}{s^2-2t^2} + \frac{-t}{s^2-2t^2}\sqrt{2} \in K$.

Exercise 7 (# 18)**.** *If $a \leq b + \epsilon$ for all $\epsilon > 0$, then $a \leq b$.*

Suppose to the contrary that $a > b$. Then choose $\epsilon = \frac{a-b}{2}$. Then $a \leq b+\epsilon = b + \frac{a-b}{2} \Rightarrow a - b \leq \frac{a-b}{2}$. This is a contradiction since $a - b \geq 0$ by hypothesis.

Exercise 8 (#23)**.** *If $a > 0$, $b > 0$, then $a < b \iff a^n < b^n$.*

We proceed by induction. The base case is obvious. For the induction step, suppose that $a < b \iff a^k < b^k$. We wish to prove that $a < b \iff a^{k+1} < b^{k+1}$.

$\Rightarrow$

If $a < b$, then by the induction hypothesis, $a^k < b^k$. Then, since $a > 0$,we have $a^{k+1} < ab^k$. Then, by Thm 2.1.7c, since $a < b$, we have $ab^k < b^{k+1}$, establishing the result.

$\Leftarrow$

Now suppose $a^{k+1} < b^{k+1}$, which is to say that $b^{k+1} - a^{k+1} > 0$. Using the axioms of the real numbers one can show that $b^{k+1} - a^{k+1} = (b-a)(b^k + a^k)$. Since $b^{k+1} - a^{k+1} > 0$ and since $b^k + a^k > 0$ It follows from the order axioms that $b - a > 0$, which is the desired result.

3 Section 2.2

Exercise 9 (# 2)**.** $|a + b| < |a| + |b| \iff ab > 0$

First notice that from the order axioms we can show that $ab > 0 \iff a > 0$ and $b > 0$ or $a < 0$ and $b < 0$.

$\Leftarrow$

There are two cases: Either $a > 0$, $b > 0$, or $a < 0, b < 0$. In the first case we have $|a+b| = a+b = |a|+|b|$ as desired. In the second case $-|a+b| = a+b = -|a| - |b|$, so multiplying both sides by -1 gives the result.

$\Rightarrow$

We prove the contrapositive, which is an equivalent statement: If $a > 0, b < 0$, then $|a+b| \neq |a|+|b|$. By definition, either $|a+b| = a+b$ or $|a+b| = -(a+b)$. In the first case, the result follows as soon as we show that $a+b \neq |a|+|b| = a-b$, which is equivalent to showing that $b \neq -b$, which follows immediately since $b \neq 0$. In the second case, we wish to show that $-(a+b) \neq |a| + |b| = a - b$, which is equivalent to showing that $a \neq -a$, which again follows since $a \neq 0$.

Exercise 10 (# 5)**.** *If $a < x < b$ and $a < y < b$ then $|x - y| < b - a$.*

Geometrically, this just says that the distance from y to x is less than the distance from b to a. Without loss of generality, let us assume that $x < y$ (if this is not so, then we can just switch the letters x and y). Then we wish to show that $y - x < b - a$. Since $y < b$ and $a < x$ it follows that $y - x < b - a$.

Exercise 11 (#14)**.** *If $\epsilon > 0$ and $\delta > 0$, then (1) $V_\epsilon(a) \cap V_\delta(a) = V_\gamma(a)$ and (2) $V_\epsilon(a) \cup V_\delta(a) = V_\gamma(a)$ for appropriate choices of γ.*

Proof. For (1), $\gamma = \min\{\epsilon, \delta\}$, for then it is clear that $|x-a| < \gamma \iff |x-a| < \delta$ and $|x-a| < \epsilon$. For (2), $\gamma = \max\{\epsilon, \delta\}$, for then it is clear that $|x-a| < \gamma \iff |x-a| < \delta$ or $|x-a| < \epsilon$. $\square$