Solutions to Homework 1- MAT319/320

September 14, 2008

Even though I haven't graded every problem of your homework, please read each solution carefully and then go back to each of your homework problems and compare your solutions to mine, correcting any problems that come up. Also, please take the time to make sure that you really understand the solutions to these problems, don't just skim over them. It may take some time, but it will help to ensure that your homework grades will steadily improve (it will improve your test grades as well). Please email me (knoxk@math.sunysb.edu) or drop by during office hours if you have any questions about these solutions. I am more than happy to discuss them with you.

1 Section 1.1

Exercise 1 (#4)**.**

(a)**.** *Show that* $D = (A \setminus B) \cup (B \setminus A)$ *(Hence the name* symmetric difference*)*

First we show that $D \subset (A \setminus B) \cup (B \setminus A)$. If $x \in D$, then there are two cases: either $x \in A$ and $x \notin A \cap B$, or $x \in B$ and $x \notin A \cap B$. We can rewrite this (why?) as: either $x \in A$ and $x \notin B$ or $x \in B$ and $x \notin A$. Thus $x \in (A \setminus B) \cup (B \setminus A)$. Next we show $(A \setminus B) \cup (B \setminus A) \subset D$. If $x \in (A \setminus B) \cup (B \setminus A)$, then either $x \in (A \setminus B)$ or $x \in (B \setminus A)$. We can rewrite this (why?) as $x \in (A \setminus A \cap B)$ or $x \in (B \setminus A \cap B)$. Thus $x \in D$.

(b)**.** *Show that* $D = (A \cup B) \setminus (A \cap B)$*.*

This follows immediately from the definition given in the statement of the problem.

Exercise 2 (#5)**.**

(a)**.** *What is* $A_1 \cap A_2$*?*

$A_1 \cap A_2 = \{n : n = 2k\} \cap \{n : n = 3k\}$ (where $k \in \mathbb{N}$, of course). But this means that 2 and 3 must be divisors of n, so we get $A_1 \cap A_2 = \{6k : k \in \mathbb{N}\} = A_5$.

(b)**.** *Determine the sets* $\bigcup\{A_n : n \in \mathbb{N}\}$ *and* $\bigcap\{A_n : n \in \mathbb{N}\}$*.*

Put $A := \bigcup\{A_n : n \in \mathbb{N}\}$ and $B := \bigcap\{A_n : n \in \mathbb{N}\}$. We claim that $A = \mathbb{N} \setminus \{1\}$ and $B = \emptyset$. If $k \in \mathbb{N}$, $k \neq 1$, then $k \in A_{k-1} \subset A$, thus $\mathbb{N} \setminus \{1\} \subset A$. It is clear that $A \subset \mathbb{N} \setminus \{1\}$. So $A = \mathbb{N} \setminus \{1\}$. To show that $B = \emptyset$, we will show that if $k \in \mathbb{N}$, then $k \notin B$. Notice that $k+1$ is the smallest element of A_k, and since $k < k+1$ it follows that $k \notin A_k$, for all $k \in \mathbb{N}$. Thus $k \notin B$.

Exercise 3 (# 8)**.**

(a)**.** *Determine $f(E)$.*

Recall from calculus that $f(x)$ is a decreasing function. So the smallest element of $f(E)$ is $f(2) = 1/4$ and the largest element is $f(1) = 1$. Thus $f(E) = [1/4, 1]$. (Note: What we have really shown is that $f(E) \subset [1/4, 1]$. The other inclusion follows from the Intermediate Value Theorem (which you may have learned in Calculus), but it is usually common to ignore this fact in "simple" problems like this.)

(b)**.** *Determine $f^{-1}(G)$.*

The inverse image is $A = [-1, -1/2] \cup [1/2, 1]$. It is easy to see that $f(A) = G$, thus $A \subset f^{-1}(G)$. On the other hand, if $x \in f^{-1}(G)$, then we have $\frac{1}{x^2} \in [1, 4]$, so $1 \leq \frac{1}{x^2} \leq 4$, so $1/4 \leq x^2 \leq 1$, so $1/2 \leq |x| \leq 1$, so $x \in A$ as desired. (Note: It really is necessary to check both inclusions, otherwise you might make the mistake of thinking that $f^{-1}(G) = [1/2, 1]$).

Exercise 4 (#18)**.**

Notice that we only need to consider f as a function $f : D(f) \to R(f)$, so f is automatically a surjection. Since it is assumed that f is an injection, it makes sense to talk about the inverse function f^{-1}.

(a)**.**

By definition, $f^{-1} = \{(f(x), x) : x \in D(f)\}$. Thus $f^{-1} \circ f(x) = x$. If $y \in R(f)$, then $y = f(x_0)$ for some $x_0 \in D(f)$. Then $f \circ f^{-1}(y) = f(f^{-1}(f(x_0))) = f(x_0) = y$ as desired.

(b)**.** *If $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is a bijection.*

Injection:
If $y_1 \neq y_2$ we want to show that $f^{-1}(y_1) \neq f^{-1}(y_2)$. Since f is surjective, write $y_1 = f(x_1)$, $y_2 = f(x_2)$. Since f is a well-defined function and since $y_1 \neq y_2$, it follows that $x_1 \neq x_2$. Using part a, we have $f^{-1}(y_1) = x_1 \neq x_2 = f^{-1}(y_2)$.
Surjection:
If $x \in A$ we want to find $y \in B$ so that $f^{-1}(y) = x$. Put $y = f(x)$. Then, by part a, $f^{-1}(y) = f^{-1} \circ f(x) = x$.

Exercise 5 (#19)**.**

Notice that $g \circ f$ is a function from A to C.
Injective:
Suppose $x_1 \neq x_2$. Since f is injective, $f(x_1) \neq f(x_2)$. Since g is injective, $g \circ f(x_1) \neq g \circ f(x_2)$.
Surjective:
Suppose $z \in C$. Since g is surjective, there exists $y \in B$ so that $g(y) = z$. Since f is surjective, there exists $x \in A$ so that $f(x) = g(y)$. Then $g \circ f(x) = z$.

Exercise 6 (#22)**.**

Suppose $f : A \to B$. First we have to show that f is bijective, to even be able to talk about an inverse function.
Injective:
Suppose $x_1 \neq x_2$. Then $g \circ f(x_1) = x_1 \neq x_2 = g \circ f(x_2)$. Since g is a well-defined function, $f(x_1) \neq f(x_2)$.
Surjective:
Suppose $y \in B$. Put $x = g(y)$. By hypothesis, $f(x) = f \circ g(y) = y$.
Now we want to show that

$$g = \{(f(x), x) : x \in A\}$$

Here we are using the definition of inverse function given on page 8 of Bartle and Sherbert (your textbook). Since $g \circ f(x) = x$ it follows that $\{(f(x), x) : x \in A\} \subset g$. To show that $\{(f(x), x) : x \in A\} \subset g$, first notice that the argument given in the first part of this proof can also be used to prove that g is a bijection. So if $(y, x) \in g$, since g is surjective we can write $(y, x) = (f(x_0), x))$. Since g is injective we see that $x = x_0$. Thus $(y, x) = (f(x), x)$. Thus $(y, x) \in \{(f(x), x) : x \in A\}$. (Note: It really is necessary to prove both inclusions. It is not good enough to just say: "g is the inverse function since $g \circ f(x) = x$.")
Remark: This problem, together with problem 18, allows us to make a new definition of inverse function that is equivalent to the one given in your book. If $f : A \to B$ is bijective, then f^{-1} is the unique function determined by the rule $f \circ f^{-1}(y) = y$ and $f^{-1} \circ f(x) = x$. This definition is usually easier to work with.

2 Section 1.2

Exercise 7 (#2)**.**

Proof by induction:
Base case ($n = 1$): $1^3 = [\frac{1}{2}(1)(1+1)]^2$.
Induction step:
Suppose $1^3 + 2^3 + \ldots + k^3 = [\frac{1}{2}k(k+1)]^2$. We want to show that

$$1^3 + 2^3 + \ldots + k^3 + (k+1)^3 = [\frac{1}{2}(k+1)(k+2)]^2.$$

From the induction hypothesis the Left Hand Side (LHS) is

$$LHS = [\frac{1}{2}k(k+1)]^2 + (k+1)^3 = (k+1)^2[\frac{1}{4}k^2 + k + 1] = (k+1)^2[\frac{1}{4}(k+2)^2]$$

which is clearly equal to the RHS (Right Hand Side).

Exercise 8 (#16)**.**

The answer is $n = 1$ and $n \geq 5$. It is easy to verify the first 4 cases. We use induction to prove the result for $n \geq 5$.
Base case $(n = 5) : 5^2 = 25 < 32 = 2^5$.
Induction step:
Suppose $k^2 < 2^k$. We want to show that $(k+1)^2 < 2^{k+1}$. Since $k^2 < 2^k$ we see that $k^2 + 1 \leq 2^k$, so it is enough to prove that $2k < 2^k$ if $k \geq 5$. We can rewrite this as $k < 2^{k-1}$. We will prove this by induction. The base case is $n = 5$. In this case we have $5 < 2^4 = 16$. For the induction step, if $j < 2^{j-1}$, we wish to show that $j + 1 < 2^j$. Since $j < 2^{j-1}$ and $1 < 2^{j-1}$ (here $j \geq 5$) it follows that $j + 1 < 2^{j-1} + 2^{j-1} = 2^j$. Thus $2k < 2^k$. Thus (from the original induction hypothesis) $k^2 + 2k + 1 < 2^{k+1}$. This is the desired result.

Exercise 9 (#20)**.**

For the base case we need to know that $1 \leq x_1 \leq 2$ and $1 \leq x_2 \leq 2$. (Why isn't it enough to just know it for x_1 ?) This is certainly true since $x_1 = 1$ and $x_2 = 2$. For the induction step, assume that x_k *and* x_{k-1} are both in the interval $[1, 2]$. Then it follows that $2 \leq x_{k-1} + x_k \leq 4$. But then $1 \leq \frac{1}{2}(x_k + x_{k-1}) \leq 2$, as desired.