Solutions of Midterm II

Name:

Student I.D:

Problem 1. (25 points) Is the infinite series $\sum_{n=1}^{+\infty} \frac{1}{-1+3n.\sqrt{n}}$ convergent? (If yes, you don't need to find the value of the limit).

Answer:

We have $\frac{1}{-1+3n.\sqrt{n}} = \frac{1}{n^{3/2}}.\frac{1}{3-\frac{1}{n^{3/2}}}$

Now $\frac{1}{3-\frac{1}{n^{3/2}}} \to \frac{1}{3}$ when $n \to +\infty$. Therefore the Comparison theorem for infinite series tells us that $\sum \frac{1}{-1+3n.\sqrt{n}}$ converges if and only if $\sum \frac{1}{n^{3/2}}$ converges.

Since the exponent $3/2 > 1$, we know that $\sum \frac{1}{n^{3/2}}$ converges and therefore our infinite series is convergent.

Problem 2. (30 points) What is $\lim_{x\to+\infty} \frac{7x^2+1}{\sqrt{2x+5}}$?

Answer:

As usual we factor by the dominant terms: $\frac{7x^2+1}{\sqrt{2x+5}} = \frac{x^2}{\sqrt{x}}.\frac{7+\frac{1}{x^2}}{\sqrt{2+\frac{5}{x}}}$

Now $\lim_{x\to+\infty} \frac{7+\frac{1}{x^2}}{\sqrt{2+\frac{5}{x}}} = \frac{7}{\sqrt{2}} > 0$, by the sum rule, the quotient rule and the square root rule.

But now the Comparison theorem for functions tells us that $f(x) = \frac{7x^2+1}{\sqrt{2x+5}}$ has a limit equal to $+\infty$ at $+\infty$ if and only if the limit of $g(x) = \frac{x^2}{\sqrt{x}} = x^{3/2}$ at $+\infty$ is equal to $+\infty$. Since this is the case, we just proved that $\lim_{x\to+\infty} \frac{7x^2+1}{\sqrt{2x+5}} = +\infty$.

Problem 3. (30 points) Use the definition of a limit (I mean use "ε, δ")

to prove that $\lim_{x\to 3} \frac{2x^2+4}{x-1} = 11$.

Answer:

As usual we study the quantity $\left|f(x) - L\right| = \left|\frac{2x^2+4}{x-1} - 11\right| = \left|\frac{2x^2+4-11x+11}{x-1}\right| = \left|\frac{2x-5}{x-1}\right|.\left|x-3\right|$

Let us prove the existence of a small neighborhood of 3 where the quantity $\left|\frac{2x-5}{x-1}\right|$ is bounded above by a constant. Consider the neighborhood $V=(2,4)$ of the point 3:

then $x \in V \Rightarrow 2 < x < 4 \Rightarrow 4 < 2x < 8 \Rightarrow -1 < 2x-5 < 3$ which implies that $-3 < 2x-5 < 3$, but this exactly means that $|2x-5| < 3$ (observe that we are only interested in an upper bound, not a lower bound).

Similarly, $x \in V \Rightarrow 2 < x < 4 \Rightarrow 1 < x-1 < 3 \Rightarrow 1 < \left|x-1\right| < 3 \Rightarrow \frac{1}{3} < \frac{1}{|x-1|} < 1$.

If you put things together, you get that for any $x \in V$ we have:

$\left|\frac{2x-5}{x-1}\right| < 3.$

Now given $\varepsilon > 0$, if we take $0 < \delta$ satisfying both $\delta < 1$ (because we want the δ-neighborhood of 3 to be included in V, which is the 1-neighborhood of 3) and $\delta < \frac{\varepsilon}{3}$, we will have the following: for any x such that $\left|x-3\right| < \delta$ we have that $\left|f(x)-11\right| < 3.\delta < 3.\frac{\varepsilon}{3} = \varepsilon$. Thus we proved that

$\lim_{x\to 3} \frac{2x^2+4}{x-1} = 11.$

Problem 4. (15 points) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that for any $x \in \mathbb{R}$, we have

$$\left|f(x) - f(1)\right| < 6.\sqrt{|x-1|}.$$

Show that such a function f is continuous at 1. (You will get some partial credit if you recall the definition of the continuity of a function at a point).

Answer:

For any given $\varepsilon > 0$, if we take $0 < \delta < \left(\frac{\varepsilon}{6}\right)^2$, we have the following:

$\left|x-1\right| < \delta$ implies that $\left|f(x) - f(1)\right| < 6.\sqrt{|x-1|} < 6.\sqrt{\delta} < 6.\frac{\varepsilon}{6} = \varepsilon$, but this means exactly that the function f is continuous at 1.