Correction of Midterm I

Problem 1. (25 points) Let $C > 0$ be a real number. Show that for any natural number $n \geq 1$, one has $(1 + C)^n \geq 1 + n\,C$.

Proof. Proof by induction:

Let's call $\mathcal{P}(n)$ the following proposition: $(1 + C)^n \geq 1 + n\,C$.

1. $\mathcal{P}(1)$ is true because $1 + C \geqslant 1 + C$;
2. Assume that $\mathcal{P}(n)$ is true:
   thus we assume that $(1 + C)^n \geq 1 + n\,C$. Now one has

$$\begin{aligned}(1+C)^{n+1} &= (1+C).(1+C)^n\\ &\geqslant (1+C).(1+nC)\,(\text{because}\,\mathcal{P}(n)\,\text{is true})\\ &= 1+nC+C+nC^2\\ &\geqslant 1+(n+1)C\end{aligned}$$

Therefore $\mathcal{P}(n+1)$ is true.

Conclusion: we proved by induction that the result is true for any integer $n \geqslant 1$.

□

Problem 2. (25 points) First version:

Find $\lim\,(x_n)$, where $x_n = \frac{1}{n+1}\sqrt{(1+2n)(n+3)}$.

Second version:

Find $\lim\,(x_n)$, where $x_n = \frac{1}{n+1}\sqrt{(n+2)(3n+1)}$.

Proof.

a) **First version:(detailed solution)**

For any $n \geqslant 1$, let's factor by the dominant terms under the square root:

$$\begin{aligned}x_n &= \frac{1}{n+1}\sqrt{(1+2n)(n+3)}\\ &= \frac{1}{n+1}\sqrt{n.\left(2+\frac{1}{n}\right).n\left(1+\frac{3}{n}\right)}\\ &= \frac{n}{n+1}\sqrt{\left(2+\frac{1}{n}\right)\left(1+\frac{3}{n}\right)}\\ &= \frac{n}{n(1+1/n)}\sqrt{\left(2+\frac{1}{n}\right)\left(1+\frac{3}{n}\right)}\\ &= \frac{1}{1+1/n}\sqrt{\left(2+\frac{1}{n}\right)\left(1+\frac{3}{n}\right)}\end{aligned}$$

At this point we recall that the archimedean property implies that the sequence $(1/n)$ converges to zero. By the Sum rule, the sequence $(1+1/n)$ converges to 1, the sequence

$(2 + 1/n)$ converges to 2, and the sequence $(1 + 3/n)$ converges to 1. By the product rule, the sequence $(2 + 1/n)(1 + 3/n)$ converges to 2. Since this last sequence is made of nonnegative terms we can apply the Square root rule and conclude that $\sqrt{\left(2+\frac{1}{n}\right)\left(1+\frac{3}{n}\right)}$ converges to $\sqrt{2}$. Finally the Quotient rule implies that $\frac{1}{1+1/n}$ converges to 1, and a final application of the product rule implies that (x_n) converges to $\sqrt{2}$.

b) **Second version:**
Similarly,

$$\begin{aligned} x_n &= \frac{1}{n+1}\sqrt{(n+2)(3n+1)} \\ &= \frac{n}{n+1}\sqrt{(1+2/n)(3+1/n)} \\ &= \frac{1}{1+1/n}\sqrt{(1+2/n)(3+1/n)} \end{aligned}$$

From this we deduce that (x_n) converges to $\sqrt{3}$.

□

Problem 3. (25 points) Working from the definition of the limit of a sequence, write a careful proof of the following statement: If (x_n) has a limit, then that limit is unique.

Proof. See the textbook, **theorem 3.1.4 Uniqueness of limits**.

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Problem 4. (25 points) First version:
Let $J_n = [1 - \frac{1}{n^2}, n+1]$. Determine $\bigcap_{n=1}^{\infty} J_n$.
Second version:
Let $J_n = [1-n, 1+\frac{1}{n^2}]$. Determine $\bigcap_{n=1}^{\infty} J_n$.

Proof.

1. **First version:**let's prove that $\bigcap_{n=1}^{\infty} J_n = [1,2]$. Indeed, for any $n \geqslant 1$, one has

$$1 - \frac{1}{n^2} \leqslant 1 < 2 \leqslant n+1$$

therefore for any $n \geqslant 1$ $[1,2] \subset J_n$, and thus $[1,2] \subset \bigcap_{n=1}^{\infty} J_n$. Now for the reverse inclusion, observe that any $x > 2$ is not in J_1 so it can't be in $\bigcap_{n=1}^{\infty} J_n$.It remains to show that any $x < 1$ cannot be in the intersection. Pick any $x < 1$, we will be done if we can find a natural number $n \geqslant 1$ such that $x = 1 - (1-x) < 1 - \frac{1}{n^2} < 1$, or equivalently such that $\frac{1}{n^2} < (1-x)$. But the archimedean property implies the existence of an integer n such that $n > \frac{1}{1-x}$, therefore one has $\frac{1}{n^2} \leqslant \frac{1}{n} < 1 - x$ and we are done.

2. **Second version:** Let's prove that $\bigcap_{n=1}^{\infty} J_n = [0,1]$.For the first inclusion, for any $n \geqslant 1$ one has

$$1 - n \leqslant 0 < 1 \leqslant 1 + \frac{1}{n^2}$$

Therefore we already know that $[0,1] \subset \bigcap_{n=1}^{\infty} J_n$. Any $x < 0$ is not in J_1 so it can't be in the intersection. It remains to prove that any $x > 1$ cannot be in the intersection. In order to do this it is enough to find some natural number $n \geqslant 1$ such that the following is true $1 \leqslant 1 + \frac{1}{n^2} \leqslant 1 + \frac{1}{n} < x$, but this is a consequence of the archimedean property (because there exists a natural number $n > \frac{1}{x-1}$).

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