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**Exercise 1.** In each of the following cases, let $T$ be the linear operator on $\mathbb{R}^2$ which is represented by the matrix $A$ in the standard ordered basis for $\mathbb{R}^2$, and let $U$ be the linear operator on $\mathbb{C}^2$ represented by $A$ in the standard ordered basis. Find the characteristic polynomial for $T$ and that for $U$, find the characteristic values of each operator, and for each such characteristic value $c$ find a basis for the corresponding space of characteristic vectors.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad A = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

**Solution:** In all cases, denote by $B_c$ the basis for the subspace corresponding to the characteristic value $c$.

First matrix The characteristic polynomial is $x(x - 1)$. The roots are 0 and 1. $B_0 = \{(0, 1)\}$ and $B_1 = \{(1, 0)\}$. In this case the real and complex cases are the same.

Second matrix The characteristic polynomial is $5 - 3x + x^2$ which has no real roots. The complex eigenvalues are $\frac{3}{2} + i\frac{1}{2}\sqrt{11}, \frac{3}{2} - i\frac{1}{2}\sqrt{11}, B_{\frac{3}{2} + i\frac{1}{2}\sqrt{11}} = \{(1, -\frac{1}{6} + i\frac{1}{6}\sqrt{11})\}$ and $B_{\frac{3}{2} - i\frac{1}{2}\sqrt{11}} = \{(1, -\frac{1}{6} - i\frac{1}{6}\sqrt{11})\}$

Third matrix The characteristic polynomial is $x^2 - 2x$. The roots are 0 and 2. $B_0 = \{(-1, 1)\}$ and $B_2 = \{(1, 1)\}$. In this case the real and complex cases are the same.

**Exercise 4.** Let $T$ be the linear operator on $\mathbb{R}^3$ which is represented in the standard ordered basis by the matrix

$$\begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix}$$

Prove that $T$ is diagonalizable by exhibiting a basis for $\mathbb{R}^3$, each vector of which is a characteristic vector of $T$.

**Solution:** The characteristic polynomial of $T$ is $(x+1)^2(x-3)$. The characteristic values are $-1$ and $3$. $B_{-1} = \{(0, 1, -1), (1, 0, 2)\}$ and $B_3 = \{(1, 1, 2)\}$. $B_{-1} \cup B_3$ is a basis for $\mathbb{R}^3$.

**Exercise 5.** Let

$$A = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix}$$

Is $A$ similar over the field $\mathbb{R}$ to a diagonal matrix? Is $A$ similar over the field $\mathbb{C}$ to a diagonal matrix?

**Solution:** The characteristic polynomial of $A$ is $-2 + x - 2x^2 + x^3$ which has roots
2, \( i \) and \(-i\). Therefore \( A \) is not similar over \( \mathbb{R} \) to a diagonal matrix, by theorem 2. Also by theorem 2 we get that \( A \) is similar to a diagonal matrix over \( \mathbb{C} \).

**Exercise 8.** Let \( A \) and \( B \) be \( n \times n \) matrices over the field \( F \). Prove that if \( I - AB \) is invertible, then \( I - BA \) is invertible and

\[(I - BA)^{-1} = I + B(I - AB)^{-1}A\]

**Solution:** Using the expression we are given, we get:

\[(I - BA)(I + B(I - AB)^{-1}A) = I - BA + B(I - AB)^{-1}A - BAB(I - AB)^{-1}A\]

After the \( I \) we can factor \( B \) on the left and \( A \) on the right, and we get:

\[(I - BA)(I + B(I - AB)^{-1}A) = I - B[-I + (I - AB)^{-1} - AB(I - AB)^{-1}]A =\]

\[I - B[-I + (I - AB)(I - AB)^{-1}]A = I + 0 = I\]

**Exercise 9.** Use the result of Exercise 8 to prove that, if \( A \) and \( B \) are \( n \times n \) matrices over the field \( F \), then \( AB \) and \( BA \) have the same characteristic values in \( F \).

**Solution:** We have to show that if \( x \) is a characteristic value for \( AB \) then \( x \) is a characteristic value for \( BA \) (and conversely). This is equivalent to the statement, if \( x \) is not a characteristic value for \( BA \) then it is not a characteristic value for \( AB \). We will prove this last statement.

Suppose that \( x \) is not a characteristic value for \( BA \), this means that \( \det(xI - BA) \neq 0 \). There are two cases:

**Case 1:** \( x = 0 \). In this case \( \det(-BA) \neq 0 \). But \( \det(-BA) = (-1)^n \det(B) \det(A) = (-1)^n \det(A) \det(B) = \det(-AB) = \det(xI - AB) \). Therefore \( \det(xI - AB) \neq 0 \).

**Case 2:** \( x \neq 0 \). In this case \( xI - BA = x(I - \frac{1}{x}BA) \) and \( \det(x(I - \frac{1}{x}BA)) = x^n \det(I - \frac{1}{x}BA) \neq 0 \). Therefore \( I - \frac{1}{x}BA \) is invertible, but this implies (by the previous exercise) that \( I - A\frac{1}{x}B = I - \frac{1}{x}AB \) is invertible, therefore \( \det(I - \frac{1}{x}AB) \neq 0 \), therefore \( x^n \det(I - \frac{1}{x}AB) = \det(xI - AB) \neq 0 \).

**Exercise 13.** Let \( V \) be the vector space of all functions from \( \mathbb{R} \) to \( \mathbb{R} \) that are continuous, i.e. the space of all continuous real-valued functions on the real line. Let \( T \) be the linear operator on \( V \) defined by

\[T(f) = \int_0^x f(t)dt.\]

Prove that \( T \) has no characteristic values.

**Solution:** Suppose that \( T \) has a characteristic value, i.e.

\[\int_0^x f(t)dt = cf(x)\]
for some $f$ not identically zero. Notice that if $c = 0$ then $f$ is identically zero by the mean value theorem why? Therefore $c \neq 0$. Define $x_0 = \sup x \in \mathbb{R} \mid f(x) = 0$. If $x_0 < \infty$, by continuity there exists $\epsilon > 0$ such that $f(x) > 0$ (or $f(x) < 0$) if $x_0 < x < x + \epsilon$. In that neighborhood we have the differential equation

$$f'(x) = cf(x)$$

with initial condition $f(x_0) = 0$. This equation has the solution $f(x) = Ke^{\frac{1}{c}x}$. But the initial condition implies that $K = 0$. Therefore $f(x) \equiv 0$ in the neighborhood $x_0 \leq x \leq x_0 + \epsilon$. Therefore $x_0 = \infty$ i.e. $f \equiv 0$ on the positive real axis. Analogously $f \equiv 0$ on the negative real axis. This contradicts the supposition that $f$ is a characteristic vector.