Homework Assignment 7

Exercise 2. Let $B = \{\alpha_1, \alpha_2, \alpha_3\}$ be the basis for $\mathbb{C}^3$ defined by
\[\alpha_1 = (1, 0, -1) \quad \alpha_2 = (1, 1, 1) \quad \alpha_3 = (2, 2, 0).\]

Find the dual basis of $B$.

Solution: The first element of the dual basis is the linear function $\alpha_1^*$ such that $\alpha_1^*(\alpha_1) = 1$, $\alpha_1^*(\alpha_2) = 0$ and $\alpha_1^*(\alpha_3) = 0$. To describe such a function more explicitly we need to find its values on the standard basis vectors $e_1$, $e_2$ and $e_3$. To do this express $e_1, e_2, e_3$ through $\alpha_1, \alpha_2, \alpha_3$ (refer to the solution of Exercise 1 pp. 54-55 from Homework 6). For each $i = 1, 2, 3$ you will find the numbers $a_i, b_i, c_i$ such that $e_1 = a_i\alpha_1 + b_i\alpha_2 + c_i\alpha_3$ (i.e. the coordinates of $e_i$ relative to the basis $\alpha_1, \alpha_2, \alpha_3$). Then by linearity of $\alpha_1^*$ we get that $\alpha_1^*(e_i) = a_i$. Then $\alpha_2^*(e_i) = b_i$, and $\alpha_3^*(e_i) = c_i$. This is the answer. It can also be reformulated as follows.

If $P$ is the transition matrix from the standard basis $e_1, e_2, e_3$ to $\alpha_1, \alpha_2, \alpha_3$, i.e. $(\alpha_1, \alpha_2, \alpha_3) = (e_1, e_2, e_3)P$, then $(P^{-1})^t$ is the transition matrix from the dual basis $e_1^*, e_2^*, e_3^*$ to the dual basis $\alpha_1^*, \alpha_2^*, \alpha_3^*$, i.e. $(\alpha_1^*, \alpha_2^*, \alpha_3^*) = (e_1^*, e_2^*, e_3^*)(P^{-1})^t$.

Note that this problem is basically the change of coordinates problem: e.g. the value of $\alpha_1^*$ on the vector $v \in \mathbb{C}^3$ is the first coordinate of $v$ relative to the basis $\alpha_1, \alpha_2, \alpha_3$.

Exercise 3.
If $A$ and $B$ are $n \times n$ matrices over the field $F$, show that $\text{trace}(AB) = \text{trace}(BA)$. Now show that similar matrices have the same trace.

Solution: It is easy to check that the trace of $AB$ as well as the trace of $BA$ equal the sum of all products $a_{ij}b_{ji}$ where $i, j = 1, \ldots, n$.

If $A$ is similar to $A'$ then $A' = PAP^{-1}$ for some invertible matrix $P$. Apply the above identity to the matrices $P$ and $AP^{-1}$. We get that $\text{trace}[P(.AP^{-1})] = \text{trace}[(AP^{-1})P]$. Hence, $\text{trace}(A') = \text{trace}[(AP^{-1})P] = \text{trace}(A)$.

Exercise 4. Let $V$ be the vector space of all polynomial functions $p$ from $\mathbb{R}$ to $\mathbb{R}$ that have degree 2 or less:
\[p(x) = c_0 + c_1x + c_2x^2.\]

Define three linear functionals on $V$ by
\[f_1(p) = \int_0^1 p(x)dx, \quad f_2(p) = \int_0^2 p(x)dx, \quad f_3(p) = \int_0^{-1} p(x)dx.\]

Show that $\{f_1, f_2, f_3\}$ is a basis for $V^*$ by exhibiting the basis for $V$ of which it is dual.

Solution:
We have to find a basis $\{p_1, p_2, p_3\}$ for $V$ such that $f_i(p_j) = \delta_{ij}$. Let’s try to find $p_1$; $p_1$ must be a polynomial of degree at most 2 such that
\[\int_0^1 p_1(x)dx = 1, \quad \int_0^2 p_1(x)dx = 0, \quad \int_0^{-1} p_1(x)dx = 0.\]
Let $P_1$ be the anti-derivative of $p_1$, i.e. $P'_1(x) = p_1(x)$. Then, by the fundamental theorem of calculus, the conditions on $p_1$ become

$$P_1(1) - P_1(0) = 1, \ P_1(2) - P_1(0) = 0, \ P_1(-1) - P_1(0) = 0$$

If we choose $P_1(0)$ to be equal to zero, the conditions are simply $P_1(1) = 1$, $P_1(2) = 0$, $P_1(-1) = 0$. This implies that $P_1$ has roots at 0, 2 and -1. Therefore $P_1(x) = q(x)x(x - 2)(x + 1)$ where $q$ is some polynomial. But we also require $P_1$ to have degree at most 3 (since we want the derivative to have degree at most 2). Therefore $q(x)$ should be a constant polynomial. What constant? Well, we want $P_1(1) = q(1)1(1 - 2)(1 + 1) = 1$ therefore $q(1) = q(x) = -\frac{1}{2}$. In the same way we can find $p_2$ and $p_3$.

Let $p_1(x) = -\frac{1}{2}[(x - 2)(x - 1) + x(x + 1) + x(x - 2)], \ p_2(x) = \frac{1}{6}[(x - 1)(x + 1) + x(x + 1) + x(x - 1)], \ p_3(x) = -\frac{1}{6}[(x - 1)(x - 2) + x(x - 2) + x(x - 1)]$. Then $B = \{p_1,p_2,p_3\}$ is a basis whose dual is $\{f_1,f_2,f_3\}$. To verify that it is a basis, it is enough to show an invertible matrix which maps the vectors $\{p_1,p_2,p_3\}$ onto the vectors $\{1,x,x^2\}$ (both sets expressed with coordinates in the basis $\{1,x,x^2\}$). This matrix is

$$M = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 2 & 2 & \frac{2}{3} \\ -1 & \frac{1}{2} & -\frac{1}{3} \end{pmatrix}$$

To verify that $\{f_1,f_2,f_3\}$ is the dual basis, calculate the integrals and verify that $f_i(p_j) = \delta_{ij}$.

**Exercise 8.** Let $W$ be the subspace of $\mathbb{R}^5$ that is spanned by the vectors

$$\alpha_1 = e_1 + 2e_2 + e_3, \quad \alpha_2 = e_2 + 3e_3 + 3e_4 + e_5, \quad \alpha_3 = e_1 + 4e_2 + 6e_3 + 4e_4 + e_5.$$ 

Find a basis for $W^0$.

**Solution:** The vectors $\alpha_1, \alpha_2, \alpha_3$ are linearly independent. This can be checked by row reducing the matrix whose row vectors are $\alpha_1, \alpha_2, \alpha_3$. We get the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 3 \\ 0 & 1 & 0 & -3 & -2 \\ 0 & 0 & 1 & 2 & 1 \end{pmatrix}$$

This matrix also shows that if we let $\alpha_4 = e_4$ and $\alpha_5 = e_5$, the ordered set $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ is a basis for $\mathbb{R}^5$. Let $\{f_1,f_2,f_3,f_4,\ldots,f_5\}$ be the dual basis. Then a basis for $W^0$ is $\{f_4,f_5\}$

**Exercise 9.** Let $V$ be the vector space of all $2 \times 2$ matrices over the field of real numbers and let

$$B = \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}$$
Let $W$ be the subspace of $V$ consisting of all $A$ such that $AB = 0$. Let $f$ be a linear functional on $V$ which is in the annihilator of $W$. Suppose that $f(I) = 0$ and $f(C) = 3$, where $I$ is the $2 \times 2$ identity matrix and

$$C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Find $f(B)$.

**Solution:** Notice that

$$B = \begin{pmatrix} -1 & -2 \\ -1 & -2 \end{pmatrix} + 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The first matrix of this sum is in $W$, therefore $f(B) = f(3I) = 3f(I) = 3 \cdot 0 = 0$

**Exercise 1.** Let $n$ be a positive integer and $F$ a field. Let $W$ be the set of all vectors $(x_1, \ldots, x_n)$ in $F^n$ such that $x_1 + \cdots + x_n = 0$.

a. Prove that $W^0$ consists of all linear functionals $f$ of the form

$$f(x_1, \ldots, x_n) = c \sum_{j=1}^{n} x_j$$

b. Show that the dual space $W^*$ of $W$ can be 'naturally' identified with the linear functionals $f(x_1, \ldots, x_n) = c_1 x_1 + \cdots + c_n x_n$ on $F_n$ which satisfy $c_1 + \cdots + c_n = 0$.

**Solution:** a. Let $f$ be a functional in $W^0$. Since $f$ is a linear functional it is of the form $f(x_1, \ldots, x_n) = a_1 x_1 + \cdots + a_n x_n$. But since $f(w) = 0$ for all $w \in W$ we must have that $f(1, -1, 0, \ldots, 0) = 0$, therefore $a_1 = a_2$. Analogously, $a_1 = a_3 = a_3 = \cdots = a_n$. Thus $f(x_1, \ldots, x_n) = a_1 (x_1 + \cdots + x_n)$.

Now let $f$ be a functional of the form $f(x_1, \ldots, x_n) = c \sum_{j=1}^{n} x_j$, then, by the definition of $W$ we must have $f(w) = 0$ for all $w \in W$, i.e. $f \in W^0$.

b. $W$ can be naturally identified with the linear functionals

$$f(x_1, \ldots, x_n) = c_1 x_1 + \cdots + c_n x_n$$

which satisfy $c_1 + \cdots + c_n = 0$ just by making a point $(w_1, \ldots, w_n)$ correspond to the functional $f(x_1, \ldots, x_n) = w_1 x_1 + \cdots + w_n x_n$.