Homework assignment 7

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Exercise 2. Let $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ be the basis for $\mathbb{C}^3$ defined by

$$\alpha_1 = (1, 0, -1) \quad \alpha_2 = (1, 1, 1) \quad \alpha_3 = (2, 2, 0).$$

Find the dual basis of $\mathcal{B}$.

Solution: The first element of the dual basis is the linear function α_1^* such that $\alpha_1^*(\alpha_1) = 1$, $\alpha_1^*(\alpha_2) = 0$ and $\alpha_1^*(\alpha_3) = 0$. To describe such a function more explicitly we need to find its values on the standard basis vectors e_1, e_2 and e_3. To do this express e_1, e_2, e_3 through $\alpha_1, \alpha_2, \alpha_3$ (refer to the solution of Exercise 1 pp. 54-55 from Homework 6). For each $i = 1, 2, 3$ you will find the numbers a_i, b_i, c_i such that $e_1 = a_i\alpha_1 + b_i\alpha_2 + c_i\alpha_3$ (i.e. the coordinates of e_i relative to the basis $\alpha_1, \alpha_2, \alpha_3$). Then by linearity of α_1^* we get that $\alpha_1^*(e_i) = a_i$. Then $\alpha_2^*(e_i) = b_i$, and $\alpha_3^*(e_i) = c_i$. This is the answer. It can also be reformulated as follows. If P is the transition matrix from the standard basis e_1, e_2, e_3 to $\alpha_1, \alpha_2, \alpha_3$, i.e. $(\alpha_1, \alpha_2, \alpha_3) = (e_1, e_2, e_3)P$, then $(P^{-1})^t$ is the transition matrix from the dual basis e_1^*, e_2^*, e_3^* to the dual basis $\alpha_1^*, \alpha_2^*, a_3^*$, i.e. $(\alpha_1^*, \alpha_2^*, a_3^*) = (e_1^*, e_2^*, e_3^*)(P^{-1})^t$.

Note that this problem is basically the change of coordinates problem: e.g. the value of α_1^* on the vector $v \in \mathbb{C}^3$ is the first coordinate of v relative to the basis $\alpha_1, \alpha_2, \alpha_3$.

Exercise 3.

If A and B are $n \times n$ matrices over the field F, show that trace(AB) =trace(BA). Now show that similar matrices have the same trace.

Solution: It is easy to check that the trace of AB as well as the trace of BA equal the sum of all products $a_{ij}b_{ji}$ where $i, j = 1, \dots, n$.

If A is similar to A' then $A' = PAP^{-1}$ for some invertible matrix P. Apply the above identity to the matrices P and AP^{-1}. We get that trace$[P(AP^{-1})] =$ trace$[(AP^{-1})P]$. Hence, trace(A') = trace$[(AP^{-1})P]$ =trace(A).

Exercise 4. Let V be the vector space of all polynomial functions p from $\mathbb{R}$ to $\mathbb{R}$ that have degree 2 or less:

$$p(x) = c_0 + c_1x + c_2x^2.$$

Define three linear functionals on V by

$$f_1(p) = \int_0^1 p(x)dx, \quad f_2(p) = \int_0^2 p(x)dx, \quad f_3(p) = \int_0^{-1} p(x)dx.$$

Show that $\{f_1, f_2, f_3\}$ is a basis for V^* by exhibiting the basis for V of which it is dual.

Solution:

We have to find a basis $\{p_1, p_2, p_3\}$ for V such that $f_i(p_j) = \delta_{ij}$. Let's try to find p_1; p_1 must be a polynomial of degree at most 2 such that

$$\int_0^1 p_1(x)dx = 1, \int_0^2 p_1(x)dx = 0, \int_0^{-1} p_1(x)dx = 0$$

Let P_1 be the anti-derivative of p_1, i.e. $P_1'(x) = p_1(x)$. Then, by the fundamental theorem of calculus, the conditions on p_1 become

$$P_1(1) - P_1(0) = 1,\ P_1(2) - P_1(0) = 0,\ P_1(-1) - P_1(0) = 0$$

If we choose $P_1(0)$ to be equal to zero, the conditions are simply $P_1(1) = 1,\ P_1(2) = 0,\ P_1(-1) = 0$. This implies that P_1 has roots at $0, 2$ and -1. Therefore $P_1(x) = q(x)x(x-2)(x+1)$ where q is some polynomial. But we also require P_1 to have degree at most 3 (since we want the derivative to have degree at most 2). Therefore $q(x)$ should be a constant polynomial. What constant? Well, we want $P_1(1) = q(1)1(1-2)(1+1) = 1$ therefore $q(1) = q(x) = -\frac{1}{2}$. In the same way we can find p_2 and p_3.

Let $p_1(x) = -\frac{1}{2}[(x-2)(x-1) + x(x+1) + x(x-2)]$, $p_2(x) = \frac{1}{6}[(x-1)(x+1) + x(x+1) + x(x-1)]$, $p_3(x) = -\frac{1}{6}[(x-1)(x-2) + x(x-2) + x(x-1)]$. Then $\mathcal{B} = \{p_1, p_2, p_3\}$ is a basis whose dual is $\{f_1, f_2, f_3\}$. To verify that it is a basis, it is enough to show an invertible matrix which maps the vectors $\{p_1, p_2, p_3\}$ onto the vectors $\{1, x, x^2\}$ (both sets expressed with coordinates in the basis $\{1, x, x^2\}$). This matrix is

$$M = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 2 & 2 & \frac{8}{3} \\ -1 & \frac{1}{2} & -\frac{1}{3} \end{pmatrix}$$

To verify that $\{f_1, f_2, f_3\}$ is the dual basis, calculate the integrals and verify that $f_i(p_j) = \delta_{ij}$.

Exercise 8. Let W be the subspace of $\mathbb{R}^5$ that is spanned by the vectors

$$\alpha_1 = e_1 + 2e_2 + e_3, \quad \alpha_2 = e_2 + 3e_3 + 3e_4 + e_5,$$

$$\alpha_3 = e_1 + 4e_2 + 6e_3 + 4e_4 + e_5.$$

Find a basis for W^0.
Solution: The vectors $\alpha_1, \alpha_2, \alpha_3$ are linearly independent. This can be checked by row reducing the matrix whose row vectors are $\alpha_1, \alpha_2, \alpha_3$. We get the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 3 \\ 0 & 1 & 0 & -3 & -2 \\ 0 & 0 & 1 & 2 & 1 \end{pmatrix}$$

This matrix also shows that if we let $\alpha_4 = e_4$ and $\alpha_5 = e_5$, the ordered set $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ is a basis for $\mathbb{R}^5$. Let $\{f_1, f_2, f_3, f_4, \ldots, f_5\}$ be the dual basis. Then a basis for W^0 is $\{f_4, f_5\}$
Exercise 9. Let V be the vector space of all 2×2 matrices over the field of real numbers and let

$$B = \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}$$

Let W be the subspace of V consisting of all A such that $AB = 0$. Let f be a linear functional on V which is in the annihilator of W. Suppose that $f(I) = 0$ and $f(C) = 3$, where I is the 2×2 identity matrix and

$$C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Find $f(B)$.

Solution: Notice that

$$B = \begin{pmatrix} -1 & -2 \\ -1 & -2 \end{pmatrix} + 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The first matrix of this sum is in W, therefore $f(B) = f(3I) = 3f(I) = 3 \cdot 0 = 0$

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Exercise 1. Let n be a positive integer and F a field. Let W be the set of all vectors $(x_1, \ldots, x_n)$ in F^n such that $x_1 + \cdots + x_n = 0$.

a Prove that W^0 consists of all linear functionals f of the form

$$f(x_1, \ldots, x_n) = c \sum_{j=1}^{n} x_j$$

b Show that the dual space W^* of W can be 'naturally' identified with the linear functionals $f(x_1, \ldots, x_n) = c_1x_1 + \cdots + c_nx_n$ on F_n which satisfy $c_1 + \cdots + c_n = 0$

Solution: a. Let f be a functional in W^0. Since f is a linear functional it is of the form $f(x_1, \ldots, x_n) = a_1x_1 + \cdots + a_nx_n$. But since $f(w) = 0$ for all $w \in W$ we must have that $f(1, -1, 0, \ldots, 0) = 0$, therefore $a_1 = a_2$. Analogously, $a_1 = a_3 = a_3 = \cdots = a_n$. Thus $f(x_1, \ldots, x_n) = a_1(x_1 + \cdots + x_n)$.
Now let f be a functional of the form $f(x_1, \ldots, x_n) = c \sum_{j=1}^{n} x_j$, then, by the definition of W we must have $f(w) = 0$ for all $w \in W$, i.e. $f \in W^0$.

b. W can be naturally identified with the linear functionals

$$f(x_1, \ldots, x_n) = c_1x_1 + \cdots + c_nx_n$$

which satisfy $c_1 + \cdots + c_n = 0$ just by making a point $(w_1, \ldots, w_n)$ correspond to the functional $f(x_1, \ldots, x_n) = w_1x_1 + \cdots + w_nx_n$