Homework assignment 4

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Exercise 8. Let V be the space of 2×2 matrices over F. Find a basis $\{A_1, A_2, A_3, A_4\}$ for V such that $A_j^2 = A_j$ for each j.

Solution If we start with the canonical basis for V, namely

$$\mathfrak{B} = \left\{ B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

we notice that the first and the last elements satisfy the required condition. Therefore we only need to find other two matrices, such that the four matrices generate V. Let

$$\mathfrak{A} = \left\{ A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Notice that $A_j^2 = A_j$ and $A_2 - A_1 = B_2$ and $A_3 - A_1 = B_3$. Therefore $\mathfrak{A}$ is a basis for V

Bonus exercise 14. Let V be the set of real numbers. Regard V as a vector space over the field of *rational* numbers, with the usual operations. Prove that this vector space is *not* finite-dimensional.

Solution: By contradiction. Suppose that V is finite dimensional, this implies that V is a countable set (see the lemma below), but the set of real numbers is not countable!

Lemma 1. *A finite dimensional vector space V over the rational numbers is a countable set.*

Proof. By induction on $n =$ the number of elements in a basis for V.
Base: $n = 1$ Let $\{\alpha\}$ be a basis for V.

Let $\{a_1, a_2, \ldots, a_n, \ldots\}$ be an enumeration of the rational numbers, then any element of V may be written as $a_i\alpha$. That is, $V \subset \{a_1\alpha, a_2\alpha, \ldots, a_n\alpha, \ldots\}$ (actually the two sets are equal, but we don't need that fact). Therefore V is countable.
Inductive Step: Suppose that V is countable if its dimension is less than or equal to n. We will prove then that *if the dimension of V is $n+1$ then it is a countable set.* Let $\{\beta_1, \ldots, \beta_n, \alpha\}$ be a basis for V. By the induction hypothesis we know that the subspace W generated by $\{\beta_1, \ldots, \beta_n\}$ is a countable set. Let

$\{w_1, w_2, \ldots, w_n, \ldots\}$ be an enumeration for W. Consider the following infinite array:

$$
\begin{array}{cccccc}
(w_1 + a_1\alpha)_1 & (w_1 + a_2\alpha)_2 & (w_1 + a_3\alpha)_4 & \cdots & (w_1 + a_n\alpha) & \cdots \\
(w_2 + a_1\alpha)_3 & (w_2 + a_2\alpha)_5 & (w_2 + a_3\alpha)_8 & \cdots & (w_2 + a_n\alpha) & \cdots \\
(w_3 + a_1\alpha)_6 & (w_3 + a_2\alpha)_9 & (w_3 + a_3\alpha)_{13} & \cdots & (w_3 + a_n\alpha) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
(w_m + a_1\alpha) & (w_m + a_2\alpha) & (w_m + a_3\alpha) & \cdots & (w_m + a_n\alpha) & \cdots \\
\vdots & \vdots & \vdots & \vdots & & \vdots
\end{array}
$$

Following the diagonals we obtain an enumeration of the array. Notice that any element $v \in V$ may be expressed as $v = a_{i_1}\beta_1 + \ldots + a_{i_n}\beta_n + a_{i_{n+1}}\alpha$ but the first n summands of the right hand side of the equality equal some $w_j \in W$ therefore we may rewrite the equation as $v = w_j + a_{i_{n+1}}\alpha$, but this element is contained in the array, i.e. V is contained in a countable set, therefore it is countable. $\square$

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Exercise 1. Show that the vectors

$$
\begin{array}{ll}
\alpha_1 = (1,1,0,0) & \alpha_2 = (0,0,1,1) \\
\alpha_3 = (1,0,0,4) & \alpha_4 = (0,0,0,2).
\end{array}
$$

form a basis in $\mathbb{R}^4$. Find the coordinates of each of the standard basis vectors in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

Solution: Write the 4×4 matrix A whose columns are $4-$tuples $\alpha_1, \alpha_2, \alpha_3$ and α_4. Check that this matrix is row (or column) equivalent to the identity matrix. Then its columns are linearly independent which means that $\alpha_1, \alpha_2, \alpha_3$ and α_4 form a basis in $\mathbb{R}^4$ (since any n linearly independent vectors in an n-dimensional space form a basis). Note that the matrix A is the transition matrix from the standard basis $e_1 = (1,0,0,0), e_2 = (0,1,0,0), e_3 = (0,0,1,0), e_4 = (0,0,0,1)$ to the basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, i.e. its i-th column gives coordinates of the vector α_i relative to the standard basis. Or in the matrix form:

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (e_1, e_2, e_3, e_4)A.$$

Multiplying both sides by A^{-1} from the right, we get that the inverse matrix A^{-1} is the transition matrix from the basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ to the standard basis, i.e. its

i-th column gives coordinates of the vector e_i relative to the basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. It remains to compute A^{-1}.

Exercise 2. Find the coordinate matrix of the vector $v = (1, 0, 1)$ in the basis $\mathcal{B}$ of $\mathbb{C}^3$ consisting of the vectors $(2i, 1, 0), (2, -1, 1), (0, 1+i, 1-i)$ in that order.

Solution: Let $\mathcal{A}$ denote the canonical basis for $\mathbb{C}^3$. Then we know that $[v]_{\mathcal{B}} = P[v]_{\mathcal{A}}$ where the columns of P are given by the coordinates in $\mathcal{B}$ of the elements in $\mathcal{A}$. Therefore

$$P^{-1} = \begin{pmatrix} 2i & 2 & 0 \\ 1 & -1 & 1+i \\ 0 & 1 & 1-i \end{pmatrix}, P = \begin{pmatrix} \frac{1}{2} - \frac{1}{2}i & -i & -1 \\ -\frac{1}{2}i & -1 & i \\ \frac{1}{4} + \frac{1}{4}i & \frac{1}{2} + \frac{1}{2}i & 1 \end{pmatrix}$$

and thus, the coordinates of v are $(-\frac{1}{2}(1+i), \frac{1}{2}i, \frac{1}{4}(3+i))$.

Exercise 4. Let W be the subspace of $\mathbb{C}^3$ spanned by $\alpha_1 = (1, 0, i)$ and $\alpha_2 = (1+i, 1, -1)$.

(a) Show that α_1 and α_2 form a basis for W.
(b) Show that the vectors $\beta_1 = (1, 1, 0)$ and $\beta_2 = (1, i, 1+i)$ are in W and form another basis for W
(c) What are the coordinates of α_1 and α_2 in the ordered basis $\{\beta_1, \beta_2\}$?

Solution 1: (a) Form the 3×2 matrix $A(\alpha)$ whose columns are triples α_1 and α_2. Check that its row (or column) reduced form does contain two nonzero rows (or columns). This proves that α_1, α_2 are linearly independent. Hence, they form a basis for W.

(b)(c) To show that β_1, β_2 are linearly independent repeat the argument of part (a).

The vector β_1 lies in W if and only if there exist scalars x_1, x_2 such that $x_1\alpha_1 + x_2\alpha_2 = \beta_1$. In other words, the system of 3 equations in 2 unknowns

$$A(\alpha) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \beta_1 \tag{1}$$

is consistent (here the triple β_1 is written as column). The same is for β_2. Hence, to show that system (1) and the corresponding system for β_2 are consistent, and to find their solutions one needs to row reduce a 3×4 matrix whose columns are triples $\alpha_1, a_2, \beta_1, \beta_2$. Then the coefficients of the row reduced matrix give the solutions of the systems, i.e. the coordinates of β_1, β_2 relative to the basis α_1, α_2. Conversely, to find the coordinates of α_1, α_2 relative to the basis β_1, β_2 row reduce the 3×4 matrix whose columns are triples $\beta_1, \beta_2, \alpha_1, a_2$ (or find the inverse of the 2×2 transition matrix from α_1, α_2 to β_1, β_2).

Solution 2: Row reducing the matrix with α_1 and α_2 and $(0, 0, 1)$ as rows, we can check that they are linearly independent, this proves (a). Notice that the solution of (c) implies (b). Let $\mathcal{B}_1$ be the basis $\{\alpha_1, \alpha_2, (0, 0, 1)\}$ for $\mathbb{C}^3$ and let

$\mathcal{B}_2 = \{\beta_1, \beta_2, (0,0,1)\}$ be another basis for $\mathbb{C}^3$. Notice that a change of basis from $\mathcal{B}_1$ to $\mathcal{B}_2$ will map W into W, i.e. it will induce the change of bases we want. Let $\mathcal{A}$ denote the canonical basis for $\mathbb{C}^3$. Then we have the following equations:

$$[v]_{\mathcal{B}_1} = P[v]_{\mathcal{B}_2}$$

$$[v]_{\mathcal{B}_1} = S[v]_{\mathcal{A}}$$

$$[v]_{\mathcal{B}_2} = Q[v]_{\mathcal{A}}$$

These equations imply the following equation $[v]_{\mathcal{B}_1} = SQ^{-1}[v]_{\mathcal{B}_2}$. Calculating we obtain:

$$S = \begin{pmatrix} 1 & -1-i & 0 \\ 0 & 1 & 0 \\ -i & i & 1 \end{pmatrix}, Q^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & i & 0 \\ 0 & 1+i & 1 \end{pmatrix}, SQ^{-1} = \begin{pmatrix} -i & 2-i & 0 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore the matrix which yields the change of bases we want is

$$\begin{pmatrix} -i & 2-i \\ 1 & i \end{pmatrix}$$

This means that $\beta_1 = -i \cdot \alpha_1 + 1 \cdot \alpha_2$ and $\beta_2 = (2-i) \cdot \alpha_1 + i \cdot \alpha_2$.

Exercise 7. Let V be the (real) vector space of all polynomial functions from $\mathbb{R}$ into $\mathbb{R}$ of degree 2 or less, i.e. the space of functions of the form

$$f(x) = c_0 + c_1 x + c_2 x^2.$$

Let t be a fixed real number and define

$$g_1(x) = 1, \quad g_2(x) = x + t, \quad g_3(x) = (x+t)^2.$$

Prove that $\mathcal{B} = \{g_1, g_2, g_3\}$ is a basis for V. If

$$f(x) = c_0 + c_1 x + c_2 x^2$$

what are the coordinates of f in the ordered basis $\mathcal{B}$?

Solution: Since

$$x^2 = ((x+t)-t)^2 = (x+t)^2 - 2t(x+t) + t^2, \quad x = x + t - t$$

we get that

$$f(x) = c_2(x+t)^2 + (c_1 - 2tc_2)(x+t) + (c_0 - tc_1 + t^2 c_2).$$

Thus, $\mathcal{B}$ span the space V, so $\mathcal{B}$ is a basis. The coordinates of f relative to $\mathcal{B}$ are $(c_2, c_1 - 2tc_2, c_0 - tc_1 + t^2 c_2)$, respectively.

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Exercise 2. Let

$$\alpha_1 = (1, 1, -2, 1), \quad \alpha_2 = (3, 0, 4, -1), \quad \alpha_3 = (-1, 2, 5, 2).$$

Let

$$\alpha = (4, -5, 9, -7), \quad \beta = (3, 1, -4, 4), \quad \gamma = (-1, 1, 0, 1).$$

(a) Which of the vectors α, β, γ are in the subspace of $\mathbb{R}^4$ spanned by the α_i?
(b) Which of the vectors α, β, γ are in the subspace of $\mathbb{C}^4$ spanned by the α_i?
(c) Does this suggest a theorem?

Solution: (a)(b) This problem is analogous to part (b) of Exercise 4 (p.55) above. Namely, consider the $4{\times}6$ matrix whose columns are vectors $\alpha_1, \alpha_2, \alpha_3, \alpha, \beta, \gamma$. Row reduce this matrix to see if the systems $x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 = \alpha (= \beta, = \gamma)$ are consistent.

(c) Theorem: Let V be a complex vector space with some basis, and $v_1, \ldots, v_n$ and v vectors with *real* coordinates with respect to this basis. Then if v is a linear combination of $v_1, \ldots, v_n$ with *complex* coefficients, then v can also be represented as a linear combination of $v_1, \ldots, v_n$ with *real* coefficients.

Exercise 3. Consider the vectors in $\mathbb{R}^4$ defined by $\alpha_1 = (-1, 0, 1, 2), \alpha_2 = (3, 4, -2, 5), \alpha_3 = (1, 4, 0, 9)$ Find a system of homogeneous linear equations for which the space of solutions is exactly the subspace of $\mathbb{R}^4$ spanned by the three given vectors.

Solution Row reducing the matrix whose rows are the α_i's we get

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & \frac{1}{4} & \frac{11}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore we need to find a matrix whose kernel is the space V spanned by $\{\alpha_1 = (1, 0, -1, -2), \alpha_2 = (0, 1, \frac{1}{4}, \frac{11}{4})\}$. The set $\mathcal{B} = \{\alpha_1, \alpha_2, (0, 0, 1, 0), (0, 0, 0, 1)\}$ is a basis for $\mathbb{R}^4$. The kernel of the matrix

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is exactly the space generated by e_1 and e_2. Thus, the matrix which maps the canonical basis to the basis $\mathcal{B}$ will map the kernel of S to the space spanned by α_1 and α_2. Let P be the change of basis, then the previous assertion expressed in terms of matrices is $P(\ker(S)) = V$ therefore $\ker(S) = P^{-1}(V)$. This means that if we apply S to any vector in $P^{-1}(V)$ we get 0, that is, if we apply P^{-1} to any vector in V and then we apply S, we get zero. But this is exactly what we want, to find a transformation whose kernel is V, such a transformation is given by SP^{-1}. Explicitly a transformations whose kernel is V is

$$SP^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & \frac{1}{4} & 1 & 0 \\ -2 & -\frac{11}{4} & 0 & 1 \end{pmatrix}$$

Exercise 6. Let V be the real vector space spanned by the rows of the matrix

$$A = \begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix}.$$

(a) Find a basis for V.

(b) Tell which vectors $v = (x_1, x_2, x_3, x_4, x_5)$ are elements of V.

(c) If $v = (x_1, x_2, x_3, x_4, x_5)$ is in V what are its coordinates in the basis chosen in part (a)?

Solution:(a)(c) Row reduce the matrix A. The nonzero rows of the row reduced matrix $\tilde{A}$ give the basis in V. It is easy to check that the coordinates of v relative to this basis form an ordered subset of coordinates x_1, x_2, x_3, x_4, x_5. This subset consists of all x_i such that i coincides with the number of a column of $\tilde{A}$ that contains the leading coefficient of a nonzero row.

(b) (The same as part (b) of Exercise 4 (p.55) above.) Take the matrix A whose columns are the basis from part (a) and the vector v. Row reduce it to write the condition on v for the system with augmented matrix A to be consistent.