homework assignment 3

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Exercise 1. Which of the following sets S of vectors $\alpha = (a_1, \ldots, a_n) \in \mathbb{R}^n$ are subspaces of $\mathbb{R}^n$ ($n \geq 3$)?

(a) all α such that $a_1 \geq 0$;	No. Take $\alpha = (1, 0, \ldots, 0) \in S$, then $(-\alpha) = (-1, 0, \ldots, n) \notin S$.
(b) all α such that $a_1 + 3a_2 = a_3$;	Yes. If $\alpha = (a_1, \ldots, a_n), \beta = (b_1, \ldots, b_n) \in S$, then $\alpha + \beta, \lambda\alpha \in S$, since $(a_1 + b_1) + 3(a_2 + b_2) =$ $= (a_1 + 3a_2) + (b_1 + 3b_2) = a_3 + b_3$ and $\lambda a_1 + 3\lambda a_2 = \lambda(a_1 + 3a_3) = \lambda a_2$.
(c) all α such that $a_2 = a_1^2$;	No. Take $\alpha = (1, 1, 0, \ldots, 0) \in S$, then $2\alpha = (2, 2, 0, \ldots, 0) \notin S$.
(d) all α such that $a_1 a_2 = 0$;	No. Take $\alpha = (0, 1, 0, \ldots, 0), \beta = (1, 0, \ldots, 0) \in S$, then $\alpha + \beta = (1, 1, 0, \ldots, 0) \notin S$.
(e) all α such that a_2 is rational.	No. Take $\alpha = (0, 1, 0, \ldots, 0)$, then $\sqrt{2}\alpha \notin S$.

Exercise 2. Let V be the (real) vector space of all functions f from $\mathbb{R}$ into $\mathbb{R}$. Which of the following sets of functions are subspaces of V?

(a) all f such that $f(x^2) = f(x)^2$;	No. Take a constant function $f(x) = 1$ for all x. Then $f \in S$, but $2f \in S$.
(b) all f such that $f(0) = f(1)$;	Yes. If $f, g \in S$, then $f + g, \lambda g \in S$, since $(f + g)(0) = f(0) + g(0) = f(1) + g(1) =$ $(f + g)(1)$, and $\lambda f(0) = \lambda f(1)$.
(c) all f such that $f(3) = 1 + f(-5)$;	No. Take a function f such that $f(3) = 1$ and $f(x) = 0$ for all $x \neq 3$. Then $f \in S$, but $2f \notin S$.
(d) all f such that $f(-1) = 0$	Yes. If $f, g \in S$, then $f + g, \lambda g \in S$, since $(f + g)(-1) = f(-1) + g(-1) = 0$ and $\lambda f(-1) = \lambda 0 = 0$.
(e) all f that are continuous.	Yes. The linear combination of continuous functions is a continuous function (although to prove it rigorously you need calculus not linear algebra).

Exercise 4. Let W be the set of all vectors $(x_1, x_2, x_3, x_4, x_5)$ in $\mathbb{R}^5$ which satisfy

$$\begin{array}{cccccc} 2x_1 & -x_2 & +\frac{4}{3}x_3 & -x_4 & = & 0 \\ x_1 & & +\frac{2}{3}x_3 & & -x_5 = & 0 \\ 9x_1 & -3x_2 & +6x_3 & -3x_4 & -3x_5 = & 0. \end{array}$$

Find a finite set of vectors that spans W.

Solution: The system of equations corresponds to the matrix

$$A = \begin{pmatrix} 2 & -1 & \frac{4}{3} & -1 & 0 \\ 1 & 0 & \frac{2}{3} & 0 & -1 \\ 9 & -3 & 6 & -3 & -3 \end{pmatrix}$$

Row reducing this matrix we obtain

$$A = \begin{pmatrix} 1 & 0 & \frac{2}{3} & 0 & -1 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which is equivalent to the conditions $x_2 = -x_4 + 2x_5$ and $x_1 = -\frac{2}{3}x_3 + x_5$. Therefore x_3,x_4 and x_5 can take any value, the conditions are only imposed on x_1 and x_2. Thus we expect 3 vectors to generate the solution space. The following vectors form a basis:

$$(-\frac{2}{3}, 0, 1, 0, 0), \quad (0, -1, 0, 1, 0), \quad (1, 2, 0, 0, 1)$$

Exercise 7. Let W_1 and W_2 be subspaces of a vector space such that the set-theoretic union of W_1 and W_2 is also a subspace. Prove that one of the spaces W_i is contained in the other.

Solution: We must show that $W_1 \nsubseteq W_2$ implies that $W_2 \subseteq W_1$ (*why?*). Suppose then that there exists a vector $w_1 \in W_1$ such that $w_1 \notin W_2$. Now let w_2 be any vector in W_2. We know that $w_1 + w_2 \in W_1 \cup W_2$ since $W_1 \cup W_2$ is a subspace, but this means that either

1. $w_1 + w_2 \in W_1$ or

2. $w_1 + w_2 \in W_2$

but the second statement leads to a contradiction: *Since* $-w_2 \in W_2$ *we must have* $(w_1 + w_2) - w_2 \in W_2$ *i.e.* $w_1 \in W_2$ *which is impossible*. Therefore the first statement must be true, but then, since $-w_1 \in W_1$ we must have $(w_1 + w_2) - w_1 \in W_1$ i.e. $w_2 \in W_1$. Hence $W_2 \subset W_1$

Exercise 8. Let V be the vector space of all functions from $\mathbb{R}$ into $\mathbb{R}$; let V_e be the subset of even functions, $f(-x) = f(x)$; let V_o be the subset of odd functions $f(-x) = -f(x)$.

(a) Prove that V_e and V_o are subspaces of V.

(b) Prove that $V_e + V_o = V$

(c) Prove that $V_e \cap V_o = \{0\}$

Solution:

(a) We will do the case of V_e (the case of V_o is analogous). V_e is not empty since the zero function $0(x) = 0$ is an element of V_e. Let f and g be two functions of V_e and let c be any real number. Then $(f + cg)(-x) = f(-x) + c \cdot g(-x) = f(x) + c \cdot g(x) = (f + cg)(x)$

(b) Let f be any function in V. Define the following functions $f_e(x) = \frac{1}{2}(f(x) + f(-x))$ and $f_o(x) = \frac{1}{2}(f(x) - f(-x))$. f_e is an even function, and f_o is an odd function (*check it!*) and we have $f = f_e + f_o$

(c) Let f be a function in $V_e \cap V_o$ and let x be any real number. Since $f \in V_e$ we must have $f(x) = f(-x)$ but since $f \in V_o$ we must have $f(x) = -f(-x)$. Therefore $f(x) = -f(x)$ but then $2f(x) = 0$ thus $f(x) = 0$. That is, f is the zero function.

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Exercise 2. Are the vectors

$$\begin{array}{cc} \alpha_1 = (1,1,2,4) & \alpha_2 = (2,-1,-5,2) \\ \alpha_3 = (1,-1,-4,0) & \alpha_4 = (2,1,1,6). \end{array}$$

linearly independent in $\mathbb{R}^4$?

Solution: No, the space they span is 2-dimensional, this is proved in exercise 3.

Exercise 3. Find a basis for the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\begin{array}{cc} \alpha_1 = (1,1,2,4) & \alpha_2 = (2,-1,-5,2) \\ \alpha_3 = (1,-1,-4,0) & \alpha_4 = (2,1,1,6). \end{array}$$

Solution: Let A be the matrix with the α_i as its row vectors, i.e.

$$A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{pmatrix}$$

Row reducing A we obtain

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore a basis for the span of the α_i's is $\{(1,0,-1,-2),(0,1,3,2)\}$.

Exercise 6. Let V be the vector space of all 2×2 matrices over the field F. Prove that V has dimension 4 by exhibiting a basis for V that has four elements.

Solution: Let

$$\mathfrak{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Exercise 7. Let V be the vector space of Exercise 6. Let W_1 be the set of matrices of the form

$$\begin{pmatrix} x & -x \\ y & z \end{pmatrix},$$

and W_2 set of matrices of the form

$$\begin{pmatrix} a & b \\ -a & c \end{pmatrix}.$$

(a) Prove that W_1 and W_2 are subspaces of V.

(b) Find the dimensions of $W_1, W_2, W_1 + W_2$, and $W_1 \cap W_2$.

Solution: The following set is a basis for W_1:

$$\mathfrak{B} = \left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Therefore W_1 has dimension 3, analogously W_2 has dimension 3. Given any 2×2 matrix

$$A = \begin{pmatrix} f & g \\ h & i \end{pmatrix}$$

we can write it as the sum of an element in W_1 and an element in W_2, namely

$$A = \begin{pmatrix} f & g \\ h & i \end{pmatrix} = \begin{pmatrix} f & -f \\ h & i \end{pmatrix} + \begin{pmatrix} 0 & g+f \\ 0 & 0 \end{pmatrix}$$

Hence, $\dim(W_1 + W_2) = 4$. Using the formula for calculating the dimension we get $\dim(W_1 \cap W_2) = 2$.

Exercise 12. Prove that the space of all $m \times n$ matrices over the field F has dimension mn by exhibiting a basis for this space.

Solution: Let A_{ij} be the $m \times n$ matrix whose entries are all zero except for the entry $\{i, j\}$ which is one. Then $\{A_{ij}\}$ is a basis for the space of all $m \times n$ matrices.