First homework assignment

p. 5

Exercise 1. Verify that the set of complex numbers of the form $x + y\sqrt{2}$, where x and y are rational, is a subfield of the field of complex numbers.

Solution: Evidently, this set contains 0 and 1. It is also easy to check that it is closed under the addition and multiplication. Then most of the axioms of the field follows directly from the corresponding properties of complex numbers. The nontrivial part of this exercise is to prove that for any element $z = x + y\sqrt{2}$, where x and y are rational, its inverse also has this form. Consider an element $z' = x - y\sqrt{2}$. Then the product $zz' = x^2 - 2y^2$ is rational. Hence, the number $\frac{z'}{x^2-2y^2}$ has the desired form. On the other hand, its product with z is 1. So the inverse is given by the formula $\frac{x}{x^2-2y^2} - \frac{y}{x^2-2y^2}\sqrt{2}$.

Exercise 3. Are the following two systems of linear equations equivalent? If so, express each equation in each system as a linear combination of the equations in the other system.

$$\begin{aligned} -x_1 + x_2 + 4x_3 &= 0 \\ x_1 + 3x_2 + 8x_3 &= 0 \\ \tfrac{1}{2}x_1 + x_2 + \tfrac{5}{2}x_3 &= 0 \end{aligned} \qquad \begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 3x_3 &= 0 \end{aligned}$$

Solution: For instance, let us express the first equation of the first system as the linear combination of the equations of the second system. Use the method of undetermined coefficients:

$$-x_1 + x_2 + 4x_3 = a(x_1 - x_3) + b(x_2 + x_3),$$

where a, b are the numbers that we want to find. Comparing coefficients with x_1 in both sides, one gets

$$-1 = a,$$

and comparing coefficients with x_2, one gets

$$1 = b.$$

Then it is easy to check that the coefficients with x_3 in both sides coincide. Hence, the first equation of the first system is the linear combination of the equations of the second system with coefficients -1 and 1. To complete the solution one just need to apply the same method and express the other equations of the first system via the equations of the second one, and vice versa.

Exercise 5 Let F be a set which contains exactly two elements, 0 and 1. Define an addition and multiplication by the tables:

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \qquad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

Verify that the set F, together with these two operations, is a field.
Solution The solution was explained in the first recitation.

Exercise 7 Prove that each subfield of the complex numbers contains every rational number.

Solution: Let F be a subfield of $\mathbb{C}$. Since F is a field it must contain a distinguished element $\tilde{0}$ which plays the role of the additive identity. But this element is unique (regarded as a complex number) therefore $\tilde{0} = 0$, where 0 is the usual 0 of the complex numbers. Therefore $0 \in F$. Analogously we may conclude that $1 \in F$. But if $1 \in F$ then we must have that $1 + 1 = 2 \in F$ (notice that $1 + 1 = 2$ since the addition inside of F is the same addition as the *usual* addition of complex numbers). Using induction we conclude that $\mathbb{N} \subset F$ (if $n \in F$ then since addition is closed $n + 1 \in F$). But F must contain all the additive inverses of all its elements, in particular, since $\mathbb{N} \subset F$ we must have $\mathbb{Z} \subset F$. Also, since F is a field, it must contain all multiplicative inverses of its elements, therefore $\frac{1}{\mathbb{Z}} := \{\frac{1}{n} \mid n \in \mathbb{Z}\} \subset F$. Now let $\frac{a}{b}$ with $a, b \in \mathbb{Z}$, $b \neq 0$ be any rational number, we know $a \in \mathbb{Z} \subset F$ and $\frac{1}{b} \in \frac{1}{Z} \subset F$ therefore $\frac{a}{b} = a \cdot \frac{1}{b} \in F$. That is, F contains every rational number.

pp. 10-11

Exercise 2. If

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix}$$

find all solutions of $AX = 0$ by row-reducing A.

Solution: Denote with $S(A, B)$ the operation of swapping rows A and B. And denote with $c \cdot A + B$ the operation of multiplying row A times c and adding it to row B.

$$\begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix} \xRightarrow{S(I,III)} \begin{pmatrix} 1 & -3 & 0 \\ 2 & 1 & 1 \\ 3 & -1 & 2 \end{pmatrix} \xRightarrow{-2\cdot I+II} \begin{pmatrix} 1 & -3 & 0 \\ 0 & 7 & 1 \\ 3 & -1 & 2 \end{pmatrix} \xRightarrow{-3\cdot I+III}$$

$$\begin{pmatrix} 1 & -3 & 0 \\ 0 & 7 & 1 \\ 0 & 8 & 2 \end{pmatrix} \xRightarrow{-\frac{1}{7}\cdot II} \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & \frac{1}{7} \\ 0 & 8 & 2 \end{pmatrix} \xRightarrow{-8\cdot II+III} \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & \frac{6}{7} \end{pmatrix} \xRightarrow{\frac{7}{6}\cdot III}$$

$$\begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore the only solution of $AX = 0$ is $X = 0$.

Exercise 3. If
$$A = \begin{pmatrix} 6 & -4 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$
find all solutions of $AX = 2X$ and all solutions of $AX = 3X$.
Solution Notice that the equation $AX = 2X$ is equivalent to the equation $AX - 2X = 0$ which is equivalent to $AX - 2IX = 0$ where I is the 3×3 identity matrix. This last equation can be factored as $(A - 2I)X = 0$. Writing explicitly $A - 2I$ we get
$$A - 2I = \begin{pmatrix} 4 & -4 & 0 \\ 4 & -4 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$
which is row equivalent to
$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$
thus the solutions are the elements in the set$\{(x_1, x_2, x_3) \mid x_1 = x_2 = x_3\}$ Analogously for the system $AX = 3X$ we get that the solutions are the elements in the set $\{(x_1, x_2, x_3) \mid x_1 = x_2 = 0\}$.

Exercise 8. Consider the system of equations $AX = 0$ where
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
is a 2×2 matrix over the field F. Prove the following.

(a) If every entry of A is 0, then every pair (x_1, x_2) is a solution of $AX = 0$.
Solution: $0 \cdot x_1 + 0 \cdot x_2 = 0$.

(b) If $ad - bc \neq 0$, then the system $AX = 0$ has only the trivial solution $x_1 = x_2 = 0$.
Solution I: Define A^{-1} in the following way:
$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -a \\ -b & c \end{pmatrix}$$
Notice that $A \cdot A^{-1} = I$ therefore, if $AX = 0$ then $AA^{-1}X = 0$, i.e. $IX = X = 0$
Solution II: Since at this point we are not supposed to know how to multiply matrices let us work a horribly complicated solution. The proof goes by cases:

1. $a=0$ in this case, since $ad-bc\neq 0$ we know that $bc\neq 0$ therefore $b\neq 0$ and $c\neq 0$. Therefore we can perform the following reduction:

$$\begin{pmatrix}0 & b\\ c & d\end{pmatrix}\xrightarrow{S(I,II)}\begin{pmatrix}c & d\\ 0 & b\end{pmatrix}\xrightarrow{\frac{1}{c}I}\begin{pmatrix}1 & \frac{d}{c}\\ 0 & b\end{pmatrix}\xrightarrow{\frac{1}{b}II}\begin{pmatrix}1 & \frac{d}{c}\\ 0 & 1\end{pmatrix}\xrightarrow{-\frac{d}{c}II+I}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

   Thus $AX=0$ and $IX=0$ have the same solutions, but the latter system only has the trivial solution.

2. $a\neq 0$ In this case we can perform the following reduction:

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\xrightarrow{\frac{1}{a}I}\begin{pmatrix}1 & \frac{b}{a}\\ c & d\end{pmatrix}\xrightarrow{-cI+II}\begin{pmatrix}1 & \frac{b}{a}\\ 0 & \frac{ad-cb}{a}\end{pmatrix}\xrightarrow{\frac{a}{ad-bc}II}\begin{pmatrix}1 & \frac{b}{a}\\ 0 & 1\end{pmatrix}\xrightarrow{-\frac{b}{a}II+I}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

   Thus $AX=0$ and $IX=0$ have the same solutions, but the latter system only has the trivial solution.

(c) If $ad-bc=0$ and some entry of A is different from 0, then there is a solution (x_1^0,x_2^0) such that (x_1,x_2) is a solution if and only if there is a scalar y such that $x_1=yx_1^0$, $x_2=yx_2^0$.

Solution:

Suppose that a is the nonzero entry. Since $ad-bc=0$ we know that $ad=bc$. Thus we can perform the following reduction of A::

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\xrightarrow{\frac{1}{a}I}\begin{pmatrix}1 & \frac{b}{a}\\ c & d\end{pmatrix}\xrightarrow{-cI+II}\begin{pmatrix}1 & \frac{b}{a}\\ 0 & 0\end{pmatrix}$$

Thus $AX=0$ if and only if $x_1=-\frac{b}{a}x_2$, letting $x_1^0=-\frac{b}{a}$ and $x_2^0=1$ we obtain what we want. An analogous argument can be given if we assume any other of the entries to be different from 0.

pp. 15-16

Exercise 3. Describe explicitly all 2×2 row-reduced echelon matrices.

Solution: The first element in the first row of a row-reduced echelon matrix is either 1 or 0. In either case the first element in the second row is 0. The second entry in the second row is again either 1 or 0. However, the former is possible only if the first entry in the first row is not 0. So there are 3 possibilities to fill out all entries except the upper right:

$$\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix},\begin{pmatrix}1 & x\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & x\\ 0 & 0\end{pmatrix}.$$

In the first case, x is necessarily 0, in the third case x is either 1 or 0, while in the second case x can be arbitrary. Thus an infinite family of 2×2 row-reduced echelon matrices parameterized by an element $c\in F$

$$\begin{pmatrix}1 & c\\ 0 & 0\end{pmatrix}$$

and three other 2×2 row-reduced echelon matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

exhaust all possibilities.

Exercise 7. Find all solutions of

$$\begin{array}{cccccc} 2x_1- & 3x_2- & 7x_3+ & 5x_4+ & 2x_5 = & -2 \\ x_1- & 2x_2- & 4x_3+ & 3x_4+ & x_5 = & -2 \\ 2x_1 & - & 4x_3+ & 2x_4+ & x_5 = & 3 \\ x_1- & 5x_2- & 7x_3+ & 6x_4+ & 2x_5 = & -7. \end{array}$$

Solution: Consider the augmented matrix of this system

$$\begin{pmatrix} 2 & -3 & -7 & 5 & 2 & -2 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 2 & 0 & -4 & 2 & 1 & 3 \\ 1 & -5 & -7 & 6 & 2 & -7 \end{pmatrix}.$$

It is easy to check that the sum of the first and the second rows is equal to the sum of the third and the fourth rows. Hence, the fourth row is the linear combination of the other rows and can be eliminated.

Now subtract the second row times 2 from the first and the second rows. We get

$$\begin{pmatrix} 0 & 1 & 1 & -1 & 0 & 2 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 0 & 4 & 4 & -4 & -1 & 7 \end{pmatrix}.$$

Interchange the first and the second rows

$$\begin{pmatrix} 1 & -2 & -4 & 3 & 1 & -2 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 4 & 4 & -4 & -1 & 7 \end{pmatrix},$$

and subtract the second row times 4 from the third one

$$\begin{pmatrix} 1 & -2 & -4 & 3 & 1 & -2 \\ 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}.$$

Consider the system corresponding to the latter matrix

$$\begin{array}{cccccc} x_1 & -2x_2 & -4x_3 & +3x_4 & +x_5 & = -2 \\ & x_2 & +x_3 & -x_4 & & = 2 \\ 0 & 0 & 0 & 0 & -x_5 & = -1 \end{array}.$$

We get that $x_5 = 1$, $x_2 = -x_3 + x_4 + 2$, $x_1 = 2x_2 + 4x_3 - 3x_4 - 3$. Hence, for any choice of (x_3, x_4) there is a unique pair (x_1, x_2) such that $(x_1, x_2, x_3, x_4, 1)$ satisfy the system. Then the solutions are parameterized by two elements $a, b \in F$:

$$\begin{aligned} x_1 &= 1 + 2a - b \\ x_2 &= 2 - a + b \\ x_3 &= a \\ x_4 &= b \\ x_5 &= 1. \end{aligned}$$

Exercise 10. Suppose R and R' are 2×3 row-reduced echelon matrices and that the systems $RX = 0$ and $R'X = 0$ have exactly the same solutions. Prove that $R = R'$.

Solution: The main idea is to prove that each row of each matrix is a linear combination of the rows of the other matrix.

First, suppose that one of the matrices R, R' has one zero row. Then the solutions of the corresponding system depend on 2 parameters. Hence, the other matrix also has a zero row, and the second rows of R, R' coincide (they are both zero). Consider a 2×3-matrix A whose first row is the first row of R and whose second row is the first row of R'. The corresponding system again has the same 2-parameter family of solutions as the systems $RX = 0, R'X = 0$. Thus a row-reduced echelon matrix equivalent to A should have one zero row. This is possible only if the rows of the matrix A coincide (since both rows of A have leading coefficients 1).

Now suppose that all rows of matrices R, R' are non-zero. Then the solutions of the corresponding system depend on 1 parameter. Form a 3×4-matrix B whose first two rows are the rows of R and whose last two rows are the rows of R'. Then B should be row equivalent to a matrix with two zero rows (otherwise the system $BX = 0$ would not have a 1-parameter family of solutions). Hence, the rows of of R' can be represented as linear combinations of the rows of R and vice versa. This is possible only if the leading coefficients of their first rows occur at the same place. Indeed, if for instance, the first row R'_1 of R' starts with more zeros than the first row of R, then so do the second row R'_2 of R' and any linear combination of R'_1, R'_2. Using similar arguments one can prove that the leading coefficients of the second rows of R, R' occur at the same place. Then it is easy to see that the only way for a row of R to be a linear combination of the rows of R' is to coincide with the respective row of R'.