There are 5 problems worth 50 points total and a bonus problem worth up to 10 points. Show all work. Always indicate carefully what you are doing in each step; otherwise it may not be possible to give you appropriate partial credit.

1. [6 points] Let \( W_1 \) and \( W_2 \) be linear subspaces of a vector space \( V \) such that \( W_1 + W_2 = V \) and \( W_1 \cap W_2 = \{0\} \). Prove that for each vector \( \alpha \in V \) there are unique vectors \( \alpha_1 \in W_1 \) and \( \alpha_2 \in W_2 \) such that \( \alpha = \alpha_1 + \alpha_2 \).

Solution: Since \( W_1 + W_2 = V \), every vector \( \alpha \in V \) can be represented as

\[
\alpha = \alpha_1 + \alpha_2
\]

for some \( \alpha_1 \in W_1 \) and \( \alpha_2 \in W_2 \).

We must then show that this representation is unique. Suppose that there are two other vectors \( \beta_1 \in W_1 \) and \( \beta_2 \in W_2 \) such that

\[
\alpha = \beta_1 + \beta_2.
\]

Then

\[
\alpha - \alpha = (\alpha_1 + \alpha_2) - (\beta_1 + \beta_2)
\]

so

\[
\alpha_1 - \beta_1 = \beta_2 - \alpha_2.
\]

Call the above vector \( \gamma \). Then since \( \alpha_1 \) and \( \beta_1 \) are both in \( W_1 \), their difference \( \gamma \) must also be in \( W_1 \). Similarly, \( \gamma \) must be in \( W_2 \), and so

\[
\gamma \in W_1 \cap W_2.
\]

Therefore \( \gamma = 0 \), that is, \( \alpha_1 = \beta_1 \) and \( \alpha_2 = \beta_2 \).
2. [12 points] Consider the vectors in \( \mathbb{R}^4 \) defined by
\[
\alpha_1 = (-1, 0, 1, 2), \quad \alpha_2 = (3, 4, -2, 5), \quad \alpha_3 = (1, 4, 0, 9).
\]

(a) [8 points] What is the dimension of the subspace \( W \) of \( \mathbb{R}^4 \) spanned by the three given vectors? Find a basis for \( W \) and extend it to a basis \( B \) of \( \mathbb{R}^4 \).

**Solution:**
\[
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{pmatrix} = \begin{pmatrix}
-1 & 0 & 1 & 2 \\
3 & 4 & -2 & 5 \\
1 & 4 & 0 & 9
\end{pmatrix} \rightarrow \begin{pmatrix}
-1 & 0 & 1 & 2 \\
0 & 4 & 1 & 11 \\
1 & 4 & 0 & 9
\end{pmatrix} \rightarrow \begin{pmatrix}
-1 & 0 & 1 & 2 \\
0 & 4 & 1 & 11 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

We can use \( \{\alpha_1, \alpha_2\} \) as a basis for \( W \) (or the first two rows of the above matrix). To extend this to a basis for \( \mathbb{R}^4 \), we can add any two vectors which are linearly independent to \( \alpha_1, \alpha_2 \), and the other vector. For example,
\[
B = \{\alpha_1, \alpha_2, (0, 0, 1, 0), (0, 0, 0, 1)\}
\]
will do nicely.

(b) [4 points] Use a basis \( B \) of \( \mathbb{R}^4 \) as in (a) to characterize all linear transformations \( T : \mathbb{R}^4 \rightarrow \mathbb{R}^4 \) that have the same null space \( W \). What can you say about the rank of such a \( T \)? What is therefore the precise condition on the values of \( T \) on \( B \)?

**Solution:** If \( T \) is a linear transformation whose null space is \( W \), then \( T(\alpha_1) = 0 \) and \( T(\alpha_2) = 0 \). If \( \beta \) is any other vector not in \( W \), then \( T(\beta) \neq 0 \) (or else \( \beta \) would be in the null space of \( T \)). Since
\[
\text{rank}(T) + \text{nullity}(T) = \text{dim}(\mathbb{R}^4) = 4
\]
and \( \text{nullity}(T) = \text{dim}(W) = 2 \), the rank of \( T \) must be 2.
3. [10 points] Let \( B = \{\alpha_1, \alpha_2, \alpha_3\} \) be the ordered basis for \( \mathbb{R}^3 \) consisting of
\[
\alpha_1 = (1, 0, -1), \quad \alpha_2 = (1, 1, 1), \quad \alpha_3 = (1, 0, 0).
\]
What are the coordinates of the vector \((a, b, c)\) in the ordered basis \(B\)?

Solution:
\[
\begin{pmatrix}
a \\
b \\
c \\
\end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},
\]

\[
a = x_1 + x_2 + x_3, \quad b = x_2, \quad c = -x_1 + x_2
\]

\[
x_1 = b - c, \quad x_2 = b, \quad x_3 = a - 2b + c
\]

Answer: \((b - c, b, a - 2b + c)\)

4. [10 points] Let \( V \) be the vector space over \( \mathbb{R} \) of all real polynomial functions \( p \) of degree at most 2. For any fixed \( a \in \mathbb{R} \) consider the shift operator \( T : V \to V \) with \((Tp)(x) = p(x + a)\).

Explain why \( T \) is linear and find the range and null space of \( T \). Is \( T \) an isomorphism? Write down the matrix of \( T \) with respect to the ordered basis \( B = \{1, x, x^2\} \).

Solution: Why \( T \) is linear:
\[
[T(p_1 + cp_2)](x) = (p_1 + cp_2)(x + a) = p_1(x + a) + cp_2(x + a) = (Tp_1)(x) + (Tp_2)(x)
\]

Null space: If \( Tp = 0 \), then \( p(x + a) = 0 \) for all \( x \). Hence, \( p(x) = 0 \) for all \( x \). So the null space is trivial.

Range: since \( \text{rk } T + \text{null } T = 3 \) and \( \text{null } T = 0 \), it follows that the range is \( V \).

Then \( T \) is an isomorphism, because its null space is trivial and its range is \( V \).

The matrix of \( T \): \( T(1) = 1, T(x) = x + a, T(x^2) = (x + a)^2 = x^2 + 2ax + a^2 \)

\[
T_B = \begin{pmatrix} 1 & a & a^2 \\ 0 & 1 & 2a \\ 0 & 0 & 1 \end{pmatrix}
\]
5. [12 points] Let $T$ be the linear operator on $\mathbb{R}^2$ defined by $T(x_1, x_2) = (-x_2, x_1)$.

(a) [3 points] What is the matrix of $T$ in the standard ordered basis for $\mathbb{R}^2$?

Solution: $T(1, 0) = (0, 1), T(0, 1) = (-1, 0)$

\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]

(b) [3 points] Interpret the operation of $T$ geometrically.

Solution: $T$ is the counterclockwise rotation by 90 degrees around the origin.

(c) [3 points] What is the matrix of $T$ in the ordered basis $B = \{\alpha_1, \alpha_2\}$, where $\alpha_1 = (1, 2)$ and $\alpha_2 = (1, -1)$?

Solution: $T(1, 2) = (-2, 1) = x_1(1, 2) + x_2(1, -1) = (x_1 + x_2, 2x_1 - x_2)$, $-1 = 3x_1$, $5 = -3x_2$

$T(1, -1) = (1, 1) = y_1(1, 2) + y_2(1, -1)$, $2 = 3y_1$, $-1 = -3y_2$

\[T_B = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}\]

(d) [3 points] Prove that for every real number $c$ the operator $(T - cI)$ is invertible.

Solution: Notice that $T^2 = -I$. Then $(T - cI)(T + cI) = T^2 - c^2I = -(1 + c^2)I$. For every real $c$ the number $1 + c^2$ is positive, thus nonzero. Hence, $-\frac{1}{1 + c^2}(T + cI)$ is the inverse of $T - cI$. 
**Bonus Problem** [up to 10 points] Let $T, U \in L(V, V)$ be linear operators on the finite dimensional vector space $V$. Prove that the rank of the composition $UT$ is less than or equal to the minimum of the ranks of $T$ and $U$.

**Solution:** (thanks to Junmeng Chen)

Note that the range of $UT$ is a subspace of the range of $U$, because $T(V) \subset V$ implies that $U(T(V)) \subset U(V)$. Hence the rank of $UT$ is less than or equal to the rank of $U$.

On the other hand, the null space of $UT$ contains the null space of $T$, because $T(\alpha) = 0$ implies that $U(T(\alpha)) = 0$. Hence the nullity of $UT$ is greater than or equal to the nullity of $T$. Since $\text{rank}(T) = \dim(V) - \text{null}(T)$ and $\text{rank}(UT) = \dim(V) - \text{null}(UT)$, it follows that the rank of $UT$ is less than or equal to the rank of $T$. 