Problem	1	2	3	4	5	**Bonus:**	**Total:**
Points	**6**	**12**	**10**	**10**	**12**	**10**	**50+10**
Scores							

Mat 310 – Linear Algebra – Fall 2004

Name: ________________________ **Id. #:** **Lecture #:**

Test 2 (November 05 / 60 minutes)

There are 5 problems worth 50 points total and a bonus problem worth up to 10 points. Show all work. Always indicate carefully what you are doing in each step; otherwise it may not be possible to give you appropriate partial credit.

1. [6 points] Let W_1 and W_2 be linear subspaces of a vector space V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$. Prove that for each vector $\alpha \in V$ there are *unique* vectors $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$ such that $\alpha = \alpha_1 + \alpha_2$.

Solution: Since $W_1 + W_2 = V$, every vector $\alpha \in V$ can be represented as

$$\alpha = \alpha_1 + \alpha_2$$

for some $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$.

We must then show that this representation is unique. Suppose that there are two other vectors $\beta_1 \in W_1$ and $\beta_2 \in W_2$ such that

$$\alpha = \beta_1 + \beta_2.$$

Then

$$\alpha - \alpha = 0 = (\alpha_1 + \alpha_2) - (\beta_1 + \beta_2)$$

so

$$\alpha_1 - \beta_1 = \beta_2 - \alpha_2.$$

Call the above vector γ. Then since α_1 and β_1 are both in W_1, their difference γ must also be in W_1. Similarly, γ must be in W_2, and so

$$\gamma \in W_1 \cap W_2.$$

Therefore $\gamma = 0$, that is, $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$.

2. [12 points] Consider the vectors in $\mathbb{R}^4$ defined by

$$\alpha_1 = (-1, 0, 1, 2), \quad \alpha_2 = (3, 4, -2, 5), \quad \alpha_3 = (1, 4, 0, 9).$$

(a) [8 points] What is the dimension of the subspace W of $\mathbb{R}^4$ spanned by the three given vectors? Find a basis for W and extend it to a basis $\mathcal{B}$ of $\mathbb{R}^4$.

Solution:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 3 & 4 & -2 & 5 \\ 1 & 4 & 0 & 9 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & 4 & 1 & 11 \\ 1 & 4 & 0 & 9 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & 4 & 1 & 11 \\ 0 & 4 & 1 & 11 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & 4 & 1 & 11 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We can use $\{\alpha_1, \alpha_2\}$ as a basis for W (or the first two rows of the above matrix). To extend this to a basis for $\mathbb{R}^4$, we can add any two vectors which are linearly independent to α_1, α_2, and the other vector. For example,

$$\mathcal{B} = \{\alpha_1, \alpha_2, (0, 0, 1, 0), (0, 0, 0, 1)\}$$

will do nicely.

(b) [4 points] Use a basis $\mathcal{B}$ of $\mathbb{R}^4$ as in (a) to characterize all linear transformations $T : \mathbb{R}^4 \to \mathbb{R}^4$ that have the same null space W. What can you say about the rank of such a T? What is therefore the precise condition on the values of T on $\mathcal{B}$?

Solution: If T is a linear transformation whose null space is W, then $T(\alpha_1) = 0$ and $T(\alpha_2) = 0$. If β is any other vector not in W, then $T(\beta) \neq 0$ (or else β would be in the null space of T). Since

$$rank(T) + nullity(T) = dim(\mathbb{R}^4) = 4$$

and $nullity(T) = dim(W) = 2$, the rank of T must be 2.

3. [10 points] Let $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ be the ordered basis for $\mathbb{R}^3$ consisting of

$$\alpha_1 = (1, 0, -1), \quad \alpha_2 = (1, 1, 1), \quad \alpha_3 = (1, 0, 0).$$

What are the coordinates of the vector (a, b, c) in the ordered basis $\mathcal{B}$?

Solution:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$a = x_1 + x_2 + x_3$, $b = x_2$, $c = -x_1 + x_2$

$x_1 = b - c$, $x_2 = b$, $x_3 = a - 2b + c$

Answer: $(b - c, b, a - 2b + c)$

4. [10 points] Let V be the vector space over $\mathbb{R}$ of all real polynomial functions p of degree at most 2. For any fixed $a \in \mathbb{R}$ consider the *shift operator* $T : V \to V$ with $(Tp)(x) = p(x + a)$. Explain why T is linear and find the range and null space of T. Is T an isomorphism? Write down the matrix of T with respect to the ordered basis $\mathcal{B} = \{1, x, x^2\}$.

Solution: Why T is linear: $[T(p_1 + cp_2)](x) = (p_1 + cp_2)(x + a) = p_1(x + a) + cp_2(x + a) = (Tp_1)(x) + (Tp_2)(x)$

Null space: If $Tp = 0$, then $p(x + a) = 0$ for all x. Hence, $p(x) = 0$ for all x. So the null space is trivial.

Range: since $\mathrm{rk}\ T + \mathrm{null}\ T = 3$ and $\mathrm{null}\ T = 0$, it follows that the range is V.

Then T is an isomorphism, because its null space is trivial and its range is V

The matrix of T: $T(1) = 1$, $T(x) = x + a$, $T(x^2) = (x + a)^2 = x^2 + 2ax + a^2$

$$T_{\mathcal{B}} = \begin{pmatrix} 1 & a & a^2 \\ 0 & 1 & 2a \\ 0 & 0 & 1 \end{pmatrix}$$

5. [12 points] Let T be the linear operator on $\mathbb{R}^2$ defined by $T(x_1, x_2) = (-x_2, x_1)$.

(a) [3 points] What is the matrix of T in the standard ordered basis for $\mathbb{R}^2$?

Solution: $T(1,0) = (0,1)$, $T(0,1) = (-1,0)$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(b) [3 points] Interpret the operation of T geometrically.

Solution: T is the counterclockwise rotation by 90 degrees around the origin.

(c) [3 points] What is the matrix of T in the ordered basis $\mathcal{B} = \{\alpha_1, \alpha_2\}$, where $\alpha_1 = (1,2)$ and $\alpha_2 = (1,-1)$?

Solution: $T(1,2) = (-2,1) = x_1(1,2) + x_2(1,-1) = (x_1 + x_2, 2x_1 - x_2)$, $-1 = 3x_1$, $5 = -3x_2$

$T(1,-1) = (1,1) = y_1(1,2) + y_2(1,-1)$, $2 = 3y_1$, $-1 = -3y_2$

$$T_{\mathcal{B}} = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{5}{3} & \frac{1}{3} \end{pmatrix}$$

(d) [3 points] Prove that for every real number c the operator $(T - cI)$ is invertible.

Solution: Notice that $T^2 = -I$. Then $(T - cI)(T + cI) = T^2 - c^2 I = -(1 + c^2)I$. For every real c the number $1 + c^2$ is positive, thus nonzero. Hence, $-\frac{1}{1+c^2}(T + cI)$ is the inverse of $T - cI$.

Bonus Problem [up to 10 points] Let $T, U \in L(V, V)$ be linear operators on the finite dimensional vector space V. Prove that the rank of the composition UT is less than or equal to the minimum of the ranks of T and U.

Solution: (thanks to Junmeng Chen)

Note that the range of UT is a subspace of the range of U, because $T(V) \subset V$ implies that $U(T(V)) \subset U(V)$. Hence the rank of UT is less than or equal to the rank of U.

On the other hand, the null space of UT contains the null space of T, because $T(\alpha) = 0$ implies that $U(T(\alpha)) = 0$. Hence the nullity of UT is greater than or equal to the nullity of T. Since $rank(T) = dim(V) - null(T)$ and $rank(UT) = dim(V) - null(UT)$, it follows that the rank of UT is less than or equal to the rank of T.