There are 5 problems worth 50 points total and a bonus problem worth up to 10 points.
Show all work. Always indicate carefully what you are doing in each step; otherwise it may not be possible to give you appropriate partial credit.

1. [15 points] Consider the homogeneous system of linear equations

\[
\begin{align*}
  x_1 + x_2 + 2x_3 - 2x_4 &= 0 \\
  x_1 - 5x_2 - x_3 + 7x_4 &= 0 \\
  x_1 - x_2 + x_3 + x_4 &= 0 
\end{align*}
\]

(a) [3 points] Write down the matrices \( A \), \( X \), and \( O \) for which the system is in matrix form \( AX = O \).

(b) [6 points] Using the Gauss-Jordan algorithm, compute the row-reduced echelon matrix \( R \) which is row equivalent to \( A \).

(c) [6 points] Use (b) to find all solutions of the above system.
2. [14 points] Let
\[ A = \begin{bmatrix} 2 & -2 & 1 \\ 3 & -1 & 2 \\ 1 & -3 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}. \]

(a) [10 points] Apply row reduction to the augmented matrix of \( A \) and determine exactly for which triples \( (y_1, y_2, y_3) \) the system \( AX = Y \) has a solution.

(b) [3 points] Write down an explicit right hand side column \( Y \) for which the system \( AX = Y \) has no solution.

3. [12 points] Consider
\[ A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}. \]

(a) [8 points] By any method, argue that \( A \) is invertible and compute \( A^{-1} \).

(b) [4 points] Write \( A \) as a product of elementary matrices.
4. [5 points] Find an example of $2 \times 2$ matrices $A, B$ for which it is not true that $(A + B)^2 = A^2 + 2AB + B^2$. [Can you give a condition for $A$ and $B$, so the last matrix formula would hold? You’ll get 5 extra points for the right answer.]

5. [5 points] Let $A$ be an $n \times n$ matrix. Show that $A$ is invertible if and only if $A^2$ is invertible.

**Bonus Problem** [up to 10 points] Consider the equation $X^2 = -I$ for $2 \times 2$ matrices. There are solutions with real coefficient. Discover as many as you can. Can you find infinitely many? Give some more additional solutions with complex coefficients. [You might also want to observe that if $P$ is invertible $2 \times 2$, then if $X$ solves, so does $PX P^{-1}$. Why?]