In addition to the examples of vector spaces discussed in the text and elsewhere, we also discussed one example in which the operations of vector addition and scalar multiplication wouldn't normally be called "addition" or "multiplication".

- Our vectors, which we denote as usual by greek letters $\alpha, \beta, \ldots$, are the positive real numbers, $\mathbb{R}^+$.
- Our field of scalars (denoted by roman letters $a, b, c, \ldots$) can be either the reals $\mathbb{R}$ or the rationals $\mathbb{Q}$. We'll write $\mathbb{Q}$ but we get a vector space with $\mathbb{R}$ as well.
- To "add" two vectors, we multiply the corresponding real numbers. That is, the vector sum of α and β is the product $\alpha\beta$.
- The scalar product of a scalar $a \in \mathbb{Q}$ and the vector $\alpha \in \mathbb{R}^+$, we compute the power α^a.

Let's check that this satisfies the necessary properties:

additive closure If α and β are positive real numbers, so is their product $\alpha\beta$.

associativity of addition Multiplication of real numbers is associative, so no problem.

commutivity of addition Multiplication of real numbers is commutative.

additive identity The positive real number 1 acts as the identity element for vector addition, since $1 \cdot \alpha = \alpha$.

additive inverses For any vector $\alpha \in \mathbb{R}^+$, there is another vector $\beta \in \mathbb{R}^+$ so when the two vectors are "added", the result is the identity element above. In this case, the inverse of α is $1/\alpha$, since $\alpha \cdot \frac{1}{\alpha} = 1$.

closure of scalar multiplication For any scalar $a \in \mathbb{Q}$ and any vector $\alpha \in \mathbb{R}^+$, the scalar multiple α^a is still in $\mathbb{R}^+$.

neutrality of 1 When we compute the scalar multiple of the multiplicative identity in our field 1 with any vector $\alpha \in \mathbb{R}^+$, we should get the original vector. That works fine: $\alpha^1 = \alpha$

vector distributive law Multiplying a scalar a times the sum of two vectors α and β:

$$(\alpha\beta)^a = \alpha^a\beta^a$$

scalar distributive law The sum of two scalars $a + b$ times a vector α:

$$\alpha^{a+b} = \alpha^a\alpha^b$$

So we see that R^+ is a vector space over $\mathbb{Q}$, with an appropriate interpretation of vector addition and scalar multiplication.

Note also that in this case, a "linear combination" works out to be very like factoring. For example, we can express the vector 40 as a linear combination of the vectors 2 and 5 by

$$40 = 2^3\, 5^1.$$

Note that there may be more than one way to express the same vector as a linear combination of two others. For example, if our underlying field is $\mathbb{R}$, then there are scalars in $\mathbb{R}$ equal to $\ln \alpha$ and $\ln \beta$, and so

$$x = \frac{\ln \beta}{\ln \alpha}$$

is also a scalar. But then if we have

$$\gamma = \alpha^a\, \beta^b,$$

we also have

$$\gamma = \alpha^{a+x}\, \beta^{b-1},$$

since

$$\alpha^x = \beta.$$

More concretely, since

$$40 = 2^3\, 5^1,$$

we also have

$$40 = 2^2\, 5^{1+\frac{\ln 2}{\ln 5}}.$$

So we can express 40 as a linear combination of 2 and 5 in many different ways. What is the dimension of $\mathbb{R}^+$ as a vector space over $\mathbb{R}$? Can you give a proof?

The situation is more complicated if we consider $\mathbb{R}^+$ as a vector space over $\mathbb{Q}$, since $x = \frac{\ln \alpha}{\ln \beta}$ will be rational for some α and β, and not for others. What do you think the dimension of this vector space is?