MATH 308 Solutions to Midterm 2

$\boxed{\text{10 pts.}}$ 1. (a) Solve the initial value problem below for $x(t)$ by any method except cheating.

$$x''(t) - x'(t) - 6x(t) = 0, \qquad x(0) = 0, \quad x'(0) = 10$$

Solution: The differential equation can be rewritten in operator form as

$$(D^2 - D - 6I)x = 0 \qquad \text{or} \qquad (D - 3I)(D + 2I)x = 0,$$

so the general form of the solution is

$$x(t) = ae^{3t} + be^{-2t}.$$

Since $x(0) = 0$, we know $a + b = 0$. Since $x'(0) = 10$, we have $3a - 2b = 10$. Consequently, $a = 2$ and $b = -2$. Thus, the solution is

$$x(t) = 2e^{3t} - 2e^{-2t}.$$

$\boxed{\text{10 pts.}}$ (b) Find the most general form of $x(t)$ for the inhomogeneous linear equation

$$x''(t) - x'(t) - 6x(t) = t$$

Solution: We need to find a particular solution x_p. Using undetermined coefficients, we look for a solution in the kernel of D^2, that is, one of the form $x_p(t) = ct + k$. Since $x_p'(t) = c$ and $x_p''(t) = 0$, we must have

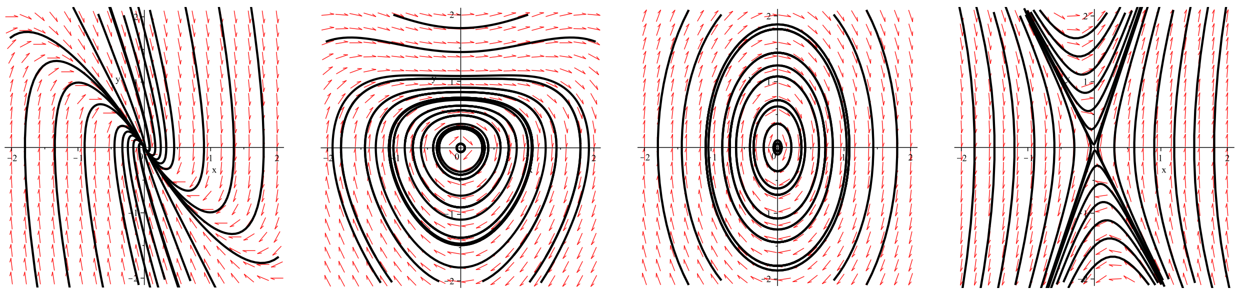
$$0 - c - 6(ct + k) = t, \quad \text{so} \quad c = -\frac{1}{6} \quad \text{and} \quad k = -\frac{c}{6} = \frac{1}{36}$$

Consequently, the general solution is the sum of the homogeneous solution from part (a) and the x_p. That is, the desired solution is

$$x(t) = ae^{3t} + be^{-2t} - \frac{t}{6} + \frac{1}{36}$$

15 pts. 2. Below are five second order differential equations labeled (a) through (e), and four phase portraits labeled 1 through 4 with a number of trajectories drawn. On the line following of each of the equations, write the letter of the corresponding phase portrait or the word "none" if the phase portrait is not shown.

1. 2. 3. 4.



(a) $x'' + 3x = 0$ (a) **3**

Solution: This has characteristic polynomial $r^2 = -3$ with eigenvalues $\pm 3i$. The real-valued solutions are of the form $A\sin(\sqrt{3}t) + B\cos(\sqrt{3}t)$. Solutions are ellipses, as in #3.

(b) $x'' - \sin(x)x' + x = 0$ (b) **2**

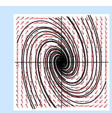
Solution: This equation is not something we've covered explicit solutions of. However, converting this to a system gives $\dot{x} = y \qquad \dot{y} = -x + \sin(x)y$ and spot checking vectors shows the phase portrait to agree with #2.

(c) $x'' + 4x' + 4x = 0$ (c) **1**

Solution: The characteristic polynomial is $(r+2)^2$, with only a single eigenvector. This matches portrait #1.

(d) $x'' + x' + 2x = 0$ (d) **none**

Solution: The characteristic polynomial has roots $\frac{-1 \pm \sqrt{7}}{2}$, and so the solutions spiral into the origin. The phase portrait is not any of the ones above.



(e) $x'' - x' - 6x = 0$ (e) **4**

Solution: This is the equation in problem 1, which has two eigenvectors, as in portrait #4.

3. Agent Orange is peacefully relaxing in his spaceship, the Defoliant, completely at rest. Suddenly, an alien battlecruiser appears 5 klicks away and applies a tractor beam which causes the Defoliant to accelerate towards it at a rate of 1 $\frac{k}{m^2}$. Fortunately, Agent Orange's countermeasures automatically kick in and are able to counteract the beam's force so that it decreases linearly to zero over the course of one minute. (Unfortunately, Agent Orange

forgot to pick up any Tylium last time he was at the store, so his engines won't start and he still drifting towards the aliens.)

If $y(t)$ is the position of the Defoliant at time t, the following differential equation holds (with an appropriate choice of units):

$$y''(t) = \begin{cases} 1-t & 0 \le t \le 1 \\ 0 & t < 0 \text{ or } t > 1 \end{cases} \qquad y(0) = y'(0) = 0.$$

10 pts. (a) Solve for $y(t)$. You may want to use the table of Laplace transforms given earlier.

Solution: We can write the equation in terms of the Heaviside function, so we have

$$y''(t) = (H_0(t) - H_1(t))(1-t) \qquad y(0) = y'(0) = 0$$

Now we apply the Laplace transform to obtain

$$s^2\mathcal{L}[y](s) - s \cdot 0 - 0 = \left(\frac{1}{s} - \frac{1}{s^2}\right) + \frac{e^{-s}}{s^2}$$

so

$$\mathcal{L}[y](s) = \frac{1}{s^3} - \frac{1}{s^4} + \frac{e^{-s}}{s^2}.$$

Now we take the inverse Laplace transform to get

$$y(t) = \frac{t^2}{2} - \frac{t^3}{6} + H_1(t)\frac{(t-1)^3}{6}.$$

This can be written without the Heaviside function as

$$y(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} - \frac{t^3}{6} & 0 \le t \le 1 \\ \frac{t}{2} - \frac{1}{6} & t > 1 \end{cases}$$

Note that if don't actually need to use the Heaviside function or Laplace transforms to do this problem. Just break the problem into three time intervals.

Before the aliens appear ($t < 0$), his position clearly satisfies $y(t) = 0$.

During the one minute the tractor beam is being applied ($0 \le t \le 1$), his position satisfies $y''(t) = 1 - t$. This is separable and can easily be solved by integrating, so $y'(t) = t - \frac{t^2}{2} + c_1$. Since $y'(0) = 0$, $c_1 = 0$. Integrating again gives $y(t) = \frac{t^2}{2} - \frac{t^3}{6} + c_2$, but since $y(0) = 0$, we know $c_2 = 0$. Hence $y(t) = \frac{t^2}{2} - \frac{t^3}{6}$ in this time interval.

After that ($t > 1$), he drifts with a constant rate equal to his velocity at $t = 1$ (which is $\frac{1}{2}$), plus the position at $t = 1$ (which is $\frac{1}{3}$). Consequently, his position for $t > 1$ is given by $\frac{t-1}{2} + \frac{1}{3}$.

Putting these three together gives the same answer as we got via Laplace transforms.

5 pts. (b) Agent Orange has a teleporter which takes exactly 10 minutes to prepare for use. If he starts preparing immediately upon sighting the aliens, does he have time to escape? (Justify your answer).

Solution: Using the answer from the previous part, we see that after 10 minutes, Agent Orange has moved $5 - \frac{1}{6}$ klicks from his original position, so he manages to escape with more than 150 meters to spare.

10 pts. 4. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$. Calculate e^{At}.

Solution: This can be done in several ways. Here's one.

The eigenvalues of A are $1, 2$, and 3, with eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 1/2 \\ 1 \\ 1 \end{pmatrix}$, respectively.

Let

$$U = \begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{and so} \qquad U^{-1} = \begin{pmatrix} 1 & -1 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, $\Lambda = U^{-1}AU$ is a diagonal matrix, and so $\Lambda_t = e^{\Lambda t}$ is easy to calculate. Now we have

$$e^{At} = U\Lambda_t U^{-1} = \begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^t & -e^t - e^{2t} & \frac{e^t - 2e^{2t} + e^{3t}}{2} \\ 0 & e^{2t} & -e^{2t} + e^{3t} \\ 0 & 0 & e^{3t} \end{pmatrix}$$

Here's another way: There are functions $a(t)$, $b(t)$ and $c(t)$ so that

$$e^t = a(t) + b(t) + c(t), \qquad e^{2t} = a(t) + 2b(t) + 2^2c(t), \qquad e^{3t} = a(t) + 3b(t) + 3^2c(t)$$

and $e^{At} = a(t)I + b(t)A + c(t)A^2$. Solving the equations above simultaneously gives

$$a(t) = 3e^t - 3e^{2t} + e^{3t}, \quad b(t) = -\frac{5}{2}e^t + 4e^{2t} - \frac{3}{2}e^{3t}, \quad c(t) = \frac{1}{2}e^t - e^{2t} + \frac{1}{2}e^{3t}$$

So, we have $e^{At} = (3e^t - 3e^{2t} + e^3t)I + (-\frac{5}{2}e^t + 4e^{2t} - \frac{3}{2}e^{3t})A + (\frac{1}{2}e^t - e^{2t} + \frac{1}{2}e^{3t})A^2$. Hence

$$\begin{aligned} e^{At} =& (3e^t - 3e^{2t} + e^3t) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{-5e^t + 8e^{2t} - 3e^{3t}}{2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \frac{e^t - 2e^{2t} + e^{3t}}{2} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{pmatrix} \\ =& \begin{pmatrix} e^t & -e^t - e^{2t} & \frac{e^t - 2e^{2t} + e^{3t}}{2} \\ 0 & e^{2t} & -e^{2t} + e^{3t} \\ 0 & 0 & e^{3t} \end{pmatrix} \end{aligned}$$

10 pts. 5. Give the general solution to the system of differential equations

$$\frac{dx}{dt} = x + y \qquad \frac{dy}{dt} = 2y + z \qquad \frac{dz}{dt} = 3z$$

Solution: This can be rewritten as the system $\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, which has the solution $e^{At}c_0$, where A is the matrix from the previous problem. So, we're kinda done already. Specifically,

$$x(t) = ae^t - b(e^t + e^{2t}) + c\frac{e^t - 2e^{2t} + e^{3t}}{2}, \quad y(t) = be^{2t} - c(e^{2t} - e^{3t}), \quad z(t) = ce^{3t}$$

Note that you can do this problem without using the results of problem 4, if you like.

First, solve the third equation to get $z(t) = ce^{3t}$.

Then the second equation becomes $y'(t) - 2y(t) = ce^{3t}$. Using undetermined coefficents, we must have a particular solution of the form $y_p = ke^{3t}$, with k satisfying $3ke^{3t} - 2e^{3t} = ce^{3t}$. This means that $k = c$. Therefore the general solution is $y(t) = Be^{2t} + ce^{3t}$.

Using the above in the first equation gives $x'(t) - x(t) = Be^{2t} + ce^{3t}$. Again using undetermined coefficients, we use a particular solution of $x_p = re^{2t} + se^{3t}$, with r and s satisfying

$$2re^{2t} + 3se^{3t} - (re^{2t} + se^{3t}) = Be^{2t} + ce^{3t}$$

Hence $r = B$ and $s = c/2$, and the general solution is $x(t) = Ae^t + Be^{2t} + \frac{c}{2}e^{3t}$.

While this may appear to be different from before, observe that if $B = b - c$ and $A = a - b + c$, we have the same solution as via the matrix exponential method. Since a, b and c are arbitrary constants, these agree.

Finally, you can do something sort of in between the two methods. Specifically, observe that eigenvalues of $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ are 1, 2, and 3, with eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 1/2 \\ 1 \\ 1 \end{pmatrix}$, respectively. Along an eigenvector v with eigenvector λ, we know that the solution will be $ce^{\lambda t}$. Because the eigenvectors are linearly independent, we now have three linearly independent solutions to the system. Since the dimension of the solution space is 3, the general solution will be the sum these three solutions. Thus, the general solution must look like

$$c_1e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3e^{3t} \begin{pmatrix} 1/2 \\ 1 \\ 1 \end{pmatrix}$$

Again, this is the same answer as before.