MAT308: Final Take-home problem

due on monday, May 23, during the final

Give a qualitative analysis of the system

$$\frac{dx}{dt} = b + y$$
$$\frac{dy}{dt} = -\sin(x) - ay$$

where a and b are real constants.

You should find and classify all equilibria, and say something about the behavior of the solutions. Where relevant, say something about eigenvalues and or eigenvectors. Pay particular attention to the bifucations that occur as a and b vary. Note that a and b can be positive, negative, or zero.

Your classification should describe what happens for all values of a and b; something along the lines of the bifurcation diagram for 2×2 systems in terms of trace and determinant is probably relevant.

You are welcome (even encouraged) to use relevant technology such as Maple, Mathematica, Matlab, or the software on the class web page to help you analyze the system. You are encouraged to include relevant pictures in your solution.

Some possibly relevant background:

This system models the behavior a damped pendulum which has a propeller attached to it, which adds a constant angular velocity b (this could be arranged so that b is in the direction of x or away from it, corresponding to $b > 0$ or $b < 0$.

Here, $x(t)$ represents the angle (in radians) that the pendulum makes with the vertical, and $y(t)$ is the angular velocity. It may help you to interpret the solutions in terms of the behaviour of the pendulum.

In class, we discussed the case $b = 0$, that is, the damped pendulum where the constant a corresponds to the amount of friction. The case $a > 0$ has "negative friction", which isn't physically realistic but still should be considered.

Recall that bifurcations occur when $a = \pm 2$ and $a = 0$ (we really only discussed the bifurcation at $a = 0$ in any detail). At $a = 2$, the spiral sink at $(0, 0)$ changes to a sink with eigenvalues $\frac{1}{2}\left(a \pm \sqrt{a^2 - 4}\right)$. In terms of the behavior of the pendulum, this means that when $0 < a < 2$, solutions which limit on $x = 0$ do so by oscillating towards straight down, while for $a > 2$ these solutions either limit directly on the equilibrium, or overshoot exactly once and then limit on the equilibrium.

Something similar happens for $a = -2$.