

# MATH 307

# Solutions to Midterm 1

- 10 pts. 1. (a) Find line of intersection between the two planes

$$5x - 3y + z = 10 \quad \text{and} \quad 2x + 4y - z = -3.$$

**Solution:** We can do this a couple of ways. I'll present two here.

First way: solve the equations simultaneously to express two variables in terms of the third; I will do this here using augmented matrices (you can use the variables if you want—it is the same thing).

$$\left( \begin{array}{ccc|c} 5 & -3 & 1 & 10 \\ 2 & 4 & -1 & -3 \end{array} \right) \rightsquigarrow \left( \begin{array}{ccc|c} 5 & -3 & 1 & 10 \\ 7 & 1 & 0 & 7 \end{array} \right) \rightsquigarrow \left( \begin{array}{ccc|c} 26 & 0 & 1 & 31 \\ 7 & 1 & 0 & 7 \end{array} \right).$$

In other words,

$$x = t, \quad y = 7 - 7t, \quad z = 31 - 26t,$$

or, in parametric form

$$\ell(t) = \langle t, 7 - 7t, 31 - 26t \rangle.$$

Alternatively, we can find the cross product of the normals of each plane, which will give a vector in the direction of the line of intersection. Then we can find a point in the intersection, and we'll have our line.

So  $\mathbf{n}_1 = \langle 5, -3, 1 \rangle$  and  $\mathbf{n}_2 = \langle 2, 3, -1 \rangle$ .  $\mathbf{n}_1 \times \mathbf{n}_2 = \langle -1, 7, 26 \rangle = \mathbf{v}$ .

Picking  $x = 0$  (we can pick anything; this one is easy) and substituting into each plane gives  $-3y + z = 10$  and  $4y - z = -3$ ; adding these gives us  $y = 7$ , and so  $z = 31$ . This means the point  $\mathbf{p} = (0, 7, 31)$  lies on the line. Now that we have a direction vector and a point, we can see that the line is given by  $\mathbf{p} + s\mathbf{v} = \langle s, 7 + 7s, 31 + 26s \rangle$ . (The parameterization is slightly different, but it is the same line.)

- 5 pts. (b) Find the cosine of the minimum angle between the two planes.

**Solution:** The normals to the two planes are  $\mathbf{n}_1 = \langle 5, -3, 1 \rangle$  and  $\mathbf{n}_2 = \langle 2, 3, -1 \rangle$ . The angle between the two planes is the angle between the normals, i.e.,

$$\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} = \frac{10 - 12 - 1}{\sqrt{25 + 9 + 1} \sqrt{4 + 16 + 1}} = \frac{-3}{\sqrt{35} \sqrt{21}}.$$

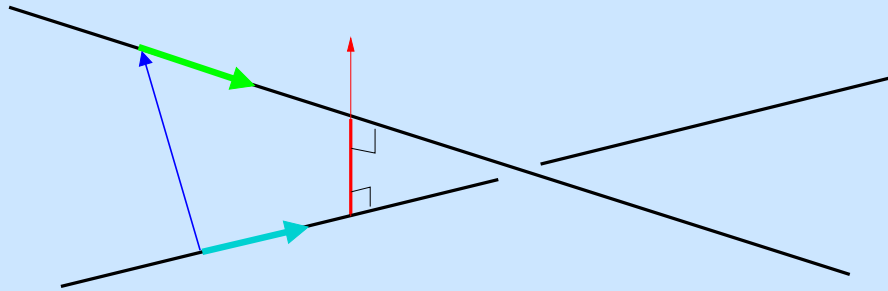
Since the cosine is negative, this is an angle larger than  $\pi/2$ , so it is the *sine* of the desired angle. This means we want  $\phi$  where  $\theta + \phi = \pi$ . So the angle we really want satisfies

$$\cos \phi = \sqrt{1 - (\sin \theta)^2} = \sqrt{1 - \frac{9}{35 \cdot 21}}.$$

- 15 pts. 2. Find the minimum distance between the two lines

$$\ell(t) = \langle 1 + t, 1 + t, 2 + t \rangle \quad \text{and} \quad \mathbf{m}(s) = \langle 3 + 2s, 1 + s, 1 + 2s \rangle.$$

**Solution:** We can visualize the situation in the following figure (just a sketch, not proper coordinates):



where the green vector is the direction of  $\ell = \langle 1, 1, 1 \rangle$ , the light blue vector is the direction of  $\mathbf{m} = \langle 2, 1, 2 \rangle$ , and we want the length of the thick red segment, which is perpendicular to both of them.

So, first we want to find the direction of the red segment, so we calculate

$$\mathbf{r} = \ell \times \mathbf{m} = \langle 1, 0, -1 \rangle.$$

Now we find a vector from one point on  $\mathbf{m}$  to a point on  $\ell$ . For example, the vector  $\mathbf{b} = \mathbf{m}(0) - \ell(0) = \langle 2, 0, -1 \rangle$  does the job. Then we calculate the length of the projection of this vector onto  $\mathbf{r}$ , that is,

$$|\text{proj}_{\mathbf{r}}(\mathbf{b})| = \frac{\mathbf{b} \cdot \mathbf{r}}{|\mathbf{r}|} = \frac{2 + 0 + 1}{\sqrt{2}} = \frac{3}{\sqrt{2}}.$$

- 10 pts. 3. (a) Find the inverse of the matrix  $A = \begin{pmatrix} 1 & 1 & 0 \\ 4 & 0 & 1 \\ 3 & 2 & 0 \end{pmatrix}$ , if it exists. If it doesn't exist, justify your answer.

**Solution:** The inverse exists, since  $\det A = 1$ . We can find the inverse via row-reduction.

$$\begin{pmatrix} 1 & 1 & 0 & | & 1 & 0 & 0 \\ 4 & 0 & 1 & | & 0 & 1 & 0 \\ 3 & 2 & 0 & | & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 & | & 1 & 0 & 0 \\ 0 & -4 & 1 & | & -4 & 1 & 0 \\ 0 & -1 & 0 & | & -3 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & | & -2 & 0 & 1 \\ 0 & 0 & 1 & | & 8 & 1 & -4 \\ 0 & 1 & 0 & | & 3 & 0 & -1 \end{pmatrix} \rightsquigarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & | & -2 & 0 & 1 \\ 0 & 1 & 0 & | & 3 & 0 & -1 \\ 0 & 0 & 1 & | & 8 & 1 & -4 \end{pmatrix} \quad \text{so} \quad A^{-1} = \begin{pmatrix} -2 & 0 & 1 \\ 3 & 0 & -1 \\ 8 & 1 & -4 \end{pmatrix}$$

5 pts.

(b) Solve the system of equations

$$x + y = u, \quad 4x + z = v, \quad 3x + 2y = w,$$

for  $x$ ,  $y$ , and  $z$  in terms of  $u$ ,  $v$ , and  $w$ . If there is no solution, justify your answer.

**Solution:** Since this system is just  $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ , we just want  $A^{-1} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ . That is,

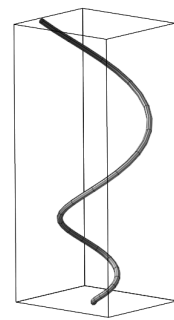
$$x = -2u + w, \quad y = 3u - w, \quad z = 8u + v - 4w.$$

15 pts.

4. Arthur Tisst is making a sculpture to put on display in the Simons Center Gallery. Art is planning to bend a platinum rod in the shape of the curve

$$\mathbf{R}(t) = \langle 3 \cos(t) + 3t \sin(t), 3 \sin(t) - 3t \cos(t), t^2 \rangle, \quad 0 \leq t \leq 4\pi.$$

What is the length of the rod ( $t$  is measured in cm) that he needs to make his sculpture?



**Solution:** We need to calculate the length of the curve  $\mathbf{R}(t)$  from  $t = 0$  to  $t = 4\pi$ . That is  $\int_0^{4\pi} |\mathbf{R}'(t)| dt$ . So, we have

$$\mathbf{R}'(t) = \langle -3 \sin(t) + (3 \sin(t) + 3t \cos(t)), 3 \cos(t) - (3 \cos(t) - 3t \sin(t)), 2t \rangle = \langle 3t \cos(t), 3t \sin(t), 2t \rangle.$$

And so

$$|\mathbf{R}'(t)| = \sqrt{9t^2 \cos^2(t) + 9t^2 \sin^2(t) + 4t^2} = \sqrt{9t^2 + 4t^2} = t\sqrt{13}.$$

This means the length is

$$\int_0^{4\pi} t\sqrt{13} dt = \frac{t^2}{2} \sqrt{13} \Big|_0^{4\pi} = 8\pi^2 \sqrt{13}.$$

- 10 pts. 5. (a) Write the equation of the plane tangent to the surface  $z = f(x, y) = 2x^2 + 4xe^{xy}$  at  $x = 1, y = 0$ .

**Solution:** Calculating the partials,

$$f_x(x, y) = 4x + 4e^{xy} + 4xy e^{xy}, \quad f_y(x, y) = 4x^2 e^{xy}.$$

So,

$$f_x(1, 0) = 4 + 4 + 0 = 8, \quad f_y(1, 0) = 4, \quad \text{and } f(1, 0) = 2 + 4 = 6.$$

Thus, the tangent plane is  $z - 6 = 8(x - 1) + 4y$ .

- 5 pts. (b) Use your answer to the above to estimate the value of  $f(1.02, -0.1)$

**Solution:** If we write the tangent plane as  $T(x, y) = 6 + 8(x - 1) + 4y$ , this question is just asking for the value of  $T(1.02, -0.1)$ . So, we have

$$f(x, y) \approx T(1.02, -0.1) = 6 + 8(1.02 - 1) + 4(-0.1) = 6 + 0.16 - 0.4 = 5.76$$

Calculating  $f(1.02, -0.1)$  to 8 places gives 5.765160571, so our estimate is not too bad.