MATH 307 Solutions to Midterm 1

$\boxed{\text{8 pts.}}$ 1. (a) Find the point of intersection of the lines

$$\langle 2+t, -1-2t, 4+t\rangle \qquad \text{and} \qquad \langle -1-2s, -1+s, -1-3s\rangle$$

Solution: We need to determine for which value of s and t the three coordinates are equal. That is, we need to solve the following linear system:

$$2+t=-1-2s \qquad -1-2t=-1+s \qquad 4+t=-1-3s$$

From the second equation, we have $s=-2t$. Replacing this in the first equation gives $2+t=-1+4t$, and so $t=1$. Thus, the first two equations give $t=1, s=-2$. We have to confirm that this is consitent with the third equation. Since $4+1=-1-3(-2)$, all is well.

The point of intersection is then obtained by plugging in to either line, giving $(3,-3,5)$ as the point where they cross.

$\boxed{\text{8 pts.}}$ (b) Write the equation of the plane containing the two lines

$$\langle 2+t, -1-2t, 4+t\rangle \qquad \text{and} \qquad \langle -1-2s, -1+s, -1-3s\rangle$$

You may express this either in vector/parametric form, or as an equation in x, y, and z. Note that this can be done even if you couldn't do the first part.

Solution: This is simplest to do in vector form. We take *any* point on the plane, and add multiples of directions parallel to the plane. Since both lines lie in the plane, any of the points $(2,-1,4)$, $(-1,-1,-1)$, or $(3,-3,5)$ will do (as will plenty of others). The directions $\langle 1,-2,1\rangle$ and $\langle -2,1,-3\rangle$ are the most obvious choices for the vectors, but a few people used others. Thus, the line

$$\langle 3,-3,5\rangle + s\langle 1,-2,1\rangle + t\langle -2,1,3\rangle$$

does the job nicely.

If you prefer the equation form of the line, a vector normal to the plane needs to be found. This can be done by taking the cross product of the direction vectors of the two lines. Since

$$\langle -2,1,3\rangle \times \langle 1,-2,1\rangle = \langle 7,5,3\rangle,$$

the equation of the plane is

$$7(x+1)+5(y+1)+3(z+1)=0, \quad \text{or} \quad 7x+5y+3z=-15.$$

However, many people added $(2,-1,4)$ and $(-1,-1,-1)$ and claimed that $(1,-2,3)$ was a good basepoint for the plane. This is not true.

$\boxed{\text{5 pts.}}$ 2. (a) Give all real values of a for which the matrix $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & a \\ 2 & a & 1 \end{pmatrix}$ has no inverse.

Solution: We can see that if $a = 0$, the first and second rows will be equal, and thus give a zero determinant (and hence a singular matrix).

Similarly, if $a = 2$, the first column and the second column are equal (or, if you prefer, the third row is the sum of the first and half of the second).

Perhaps more straightforward is to calculate the determinant of the matrix, which is $-a^2 + 2a = a(2-a)$. This is zero exactly when $a = 0$ or $a = 2$.

$\boxed{\text{5 pts.}}$ (b) Find the inverse of the matrix $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$. If the matrix has no inverse, say why not.

Solution: While a couple of people tried to use the formula for the inverse, row-reduction is probably the most straightforward method here. There are many paths up the mountain; here's one:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

Since it is so easy to make an arithmetic error, we should check that we really found the inverse:

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0-1+2 & 0-1-1 & 0-1-1 \\ 1+1-2 & 1+1-1 & 0+1-1 \\ -1+1+0 & -1+1+0 & 0+1+0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So $\begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix}$ is indeed the inverse.

$\boxed{\text{5 pts.}}$ (c) Find all solutions to the system of equations $\begin{cases} x & +y & & = 2 \\ x & +y & +z & = 1 \\ 2x & +y & +z & = 2 \end{cases}$.

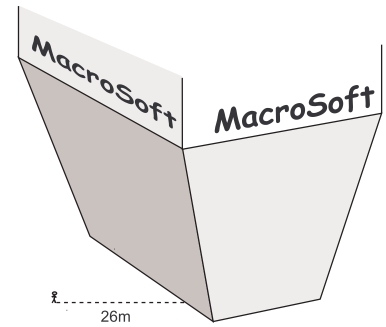
Solution: After solving the previous part, this was *supposed* to be trivial. The given system is $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, so the solution is obtained by multiplying both sides by the inverse found in the previous part. Thus

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

Most people did row-reduction *again*, essentially repeating part b. Silly rabbits.

15 pts. 3. Department of Justice Special Agent Orange, while investigating charges of anti-competitive business practices, walks due east out of the MacroSoft corporate office building for a distance of exactly 26 meters. He pauses for a moment to tie his shoe, and suddenly he finds himself surrounded by an advancing wall of flames! He can't move, and there is no ground-based escape.

However, he has his HookShot™ grappling hook with him, which he can fire at any fixed object within 22.5 meters and pull himself to it. He recalls from his careful study of the building plans that the nearest face of the building is a flat plane, inclined outwards with a normal vector of $\langle 4, 2\sqrt{2}, -1\rangle$ relative to east (that is, in the coordinate system where east is $\langle 1, 0, 0\rangle$). How far is Agent Orange from the nearest part of the building face? Can he escape, or is he toast?



Solution: This is just asking if the distance from Agent Orange's position to the plane made by the building face is less than 22.5. If we put the origin at the door, then Agent Orange is at $(26, 0, 0)$, and the plane in question contains the point $(0, 0, 0)$ and has normal $\langle 4, 2\sqrt{2}, -1\rangle$. The question is asking for the component of $V = \langle 26, 0, 0\rangle$ lying in the direction $N = \langle 4, 2\sqrt{2}, -1\rangle$.

Since $V \cdot N = |V|\,|N| \cos\theta$ and we want $|V| \cos\theta$, we calculate

$$|N| = \sqrt{16 + 8 + 1} = 5 \qquad \text{and} \qquad V \cdot N = 26 \cdot 4 + 0 + 0 = 104.$$

His distance from the face of the building is $104/5 = 20\frac{4}{5}$ meters, so he has 1.7 meters to spare, and easily pulls himself to safety.

Unfortunately, just as he made it to the building, the office occupant opened the window, and Agent Orange fell just over 4 meters to the ground, breaking his ankle in the process. Better to have a broken ankle than to become a human tiki-torch.

$\boxed{\text{8 pts.}}$ 4. (a) For the function $F : \mathbb{R}^2 \to \mathbb{R}^3$ given by $F(x,y) = \langle x\cos(xy), x\sin(y), 2y\rangle$, find the partials $F_x(x,y)$ and $F_y(x,y)$.

Solution: We just take partials of each component. Many people forgot the chain rule.

$$F_x(x,y) = \langle \cos(xy) - xy\sin(xy), \sin(y), 0\rangle \qquad F_y(x,y) = \langle -x^2\sin(xy), x\cos(y), 2\rangle$$

$\boxed{\text{8 pts.}}$ (b) Write the equation of a plane tangent to the parametric surface $\langle s\cos(st), s\sin(t), 2t\rangle$ at the point $(-1, 0, 2\pi)$.

Solution: A lot of people didn't realize that we need to determine the value of s and t for which the surface passes through the point $(-1, 0, 2\pi)$. But we do. So we find values of s and t so that $s\cos(st) = -1$, $s\sin(t) = 0$, and $2t = 2\pi$. The third equation immediately tells us that $t = \pi$, and from the first we must have $s\cos(s\pi) = -1$. While there are many values of s that satisfy this (the surface actually passes through the given point infinitely often), $s = 1$ is the obvious solution. (A couple of people claimed $s = -1$ works, but $-\cos(-\pi) = +1$, so it doesn't.)

With that in hand, the rest is easy. Observe that

$$\begin{aligned} F_x(1,\pi) &= \langle \cos(\pi) - \pi\sin(\pi), \sin(\pi), 0\rangle &&= \langle -1, 0, 0\rangle \\ F_y(1,\pi) &= \langle -\sin(\pi), \cos(\pi), 2\rangle &&= \langle 0, -1, 2\rangle \end{aligned}$$

In vector form, the plane is

$$\langle -1, 0, 2\pi\rangle + u\langle -1, 0, 0\rangle + v\langle 0, -1, 2\rangle.$$

$\boxed{\text{12 pts.}}$ 5. Listed are six functions, labeled **a** through **f**. Underneath each surface below, write the letter of the function it is the graph of.

a. $(x+y)^2$

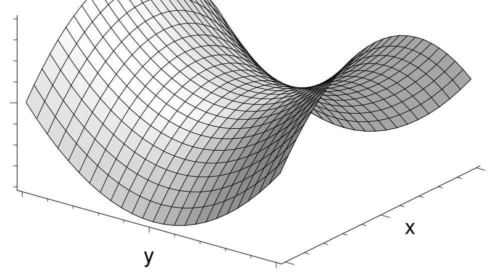
b. $\sqrt{x^2+y^2}$

c. $\cos\left(x^2+y^2\right)$

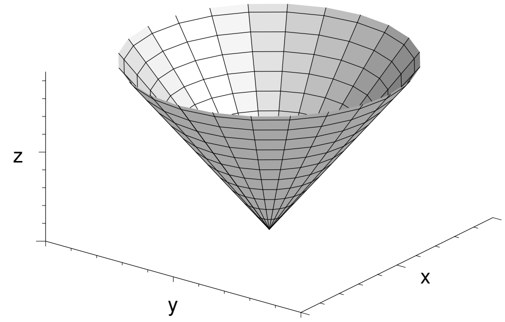
d. y^2-x^2

e. $e^{-1-x^2-y^4}$

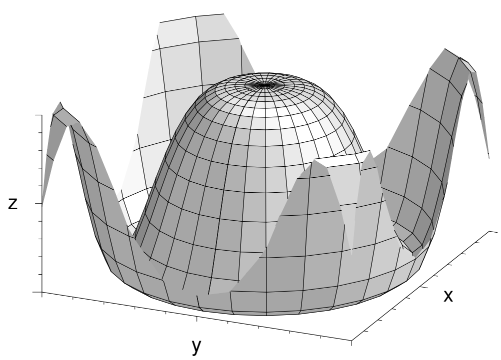
f. $\cos\left(\dfrac{(x-y)^2}{2}\right)$



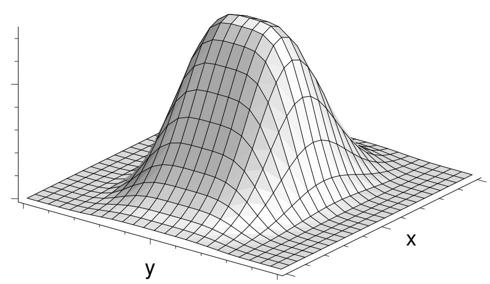
I is the graph of **d:** y^2-x^2



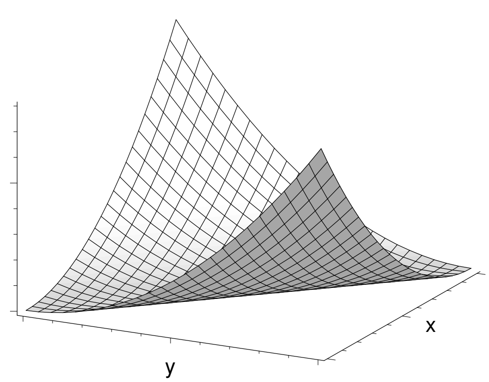
IV is the graph of **a:** $\sqrt{x^2+y^2}$



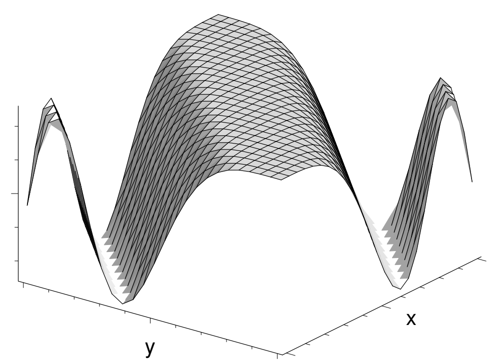
II is the graph of **c:** $\cos\left(x^2+y^2\right)$



V is the graph of **e:** $e^{-1-x^2-y^4}$



III is the graph of **a:** $(x+y)^2$



VI is the graph of **f:** $\cos\left(\dfrac{(x-y)^2}{2}\right)$