MATH 200 Solutions to Midterm 2

1. [10 points] Suppose a is a nonzero rational number, and x is an irrational number. Prove that ax is an irrational number.

Solution: For contradiction, we suppose that ax were rational. Since a is rational, there are nonzero integers p and q so that $a = p/q$; furthermore, since ax is rational, there are nonzero integers r and s with so that $ax = r/s$. Thus, we have

$$ax = \frac{p}{q}x = \frac{r}{s}.$$

Solving for x gives us

$$x = \frac{rq}{sp}.$$

Note that $qs \neq 0$ since both q and s are nonzero. (Note that $gcd(rq, ps)$ might not be 1, since either r and p or q and s might have a common divisor. However, this is irrelevant: we can just reduce the fraction.)

Thus, if $ax \in \mathbb{Q}$, we have shown that $x \in \mathbb{Q}$, a contradiction.

2. [10 points] Let A be a set of positive integers with no least element. Show that A must be the empty set.

Hint: Prove by induction on n that if A has no least element, then $\mathbb{N}_n$ is disjoint from A for every n.

Solution:

Informally, the idea of the suggestion is that we show (by induction), that $1 \notin A$, and then $2 \notin A$, and so $3 \notin A$, and so on. Thus, no positive integer can be in A.

So, we show $\mathbb{N}_n \cap A = \emptyset$ by induction on n.

For the base case, take $n = 1$. If $1 \in A$, then since 1 is the smallest positive integer, A would have a least element, contradicting the hypotheses. Thus $\mathbb{N}_1 \cap A = \emptyset$.

Now we show that if $\mathbb{N}_k \cap A = \emptyset$, then $\mathbb{N}_{k+1}$ is disjoint from A. Suppose A does not contain any element of $\mathbb{N}_k$. Since A has no elements of $\mathbb{N}_k$ in it, every element a of A satisfies $a \geq k+1$. If $k+1 \in A$, then since $k+1 \leq a$ for every $a \in A$, $k+1$ is the the least element of A. Thus, since A has no least element, $k+1 \notin A$. Therefore $\mathbb{N}_{k+1} \cap A = \emptyset$, as desired.

Thus, we have shown that for every $n \in \mathbb{Z}^+$, $\mathbb{N}_n$ is disjoint from A. But $Z^+ = \bigcup \mathbb{N}_n$, so no positive integer is an element of A. Hence A is the empty set.

Many people tried to do this by induction on $|A|$, showing that if $|A| = 1$ then A has a least element, and then that if any set of size n has a least element then so does any set of size $n + 1$. But this just shows that every finite set of integers has a least element, but doesn't handle the case where A is infinite.

3. (a) 5 points Carefully prove that if A and B are disjoint denumerable sets, then $A \cup B$ is also denumerable.

Solution: Since A and B are denumerable, we have

$$A = \{a_1, a_2, a_3, \ldots\} \qquad B = \{b_1, b_2, b_3, \ldots\},$$

that is, we have bijections $f : \mathbb{Z}^+ \to A$ and $g : \mathbb{Z}^+ \to B$. What we need is to give a way to list $A \cup B$, that is, a bijection $h : \mathbb{Z}^+ \to A \cup B$.

Note that we can't just list the elements of A followed by those of B: since A is infinite, we'll never get to B. So we take the "one for you, one for me" strategy, and alternate between the two sets, that is,

$$A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, \ldots\}.$$

More formally, we can write the bijection $h : \mathbb{Z}^+ \to A \cup B$ as

$$h(i) = \begin{cases} f\left(\frac{i+1}{2}\right) & \text{if } i \text{ is odd} \\ g\left(\frac{i}{2}\right) & \text{if } i \text{ is even} \end{cases}$$

(b) 5 points Show that if X is an uncountable set and $A \subseteq X$ is denumerable, then the complement of A in X (that is, $X - A$) must be uncountable.

You may use the first part of this question, even if you couldn't do it

Solution: We can do this by contradiction. If $X - A$ is not uncountable, then it must be countable, that is either finite or denumerable.

If $X - A$ is denumerable, we have X expressed as the union of two denumerable sets: $X = A \cup (X - A)$, and so by the first part of the problem, X is denumerable, giving a contradiction.

Similarly, if $X - A$ is finite, since A is denumerable, their union is again denumerable, giving a contradition. (There is a theorem in the text to this effect. However, the proof is simple: If $|X - A| = n$, then we can write $X - A = \{x_1, x_2, x_3, \ldots, x_n\}$, and so $X = \{x_1, x_2, x_3, \ldots, x_n, a_1, a_2, a_3, \ldots\}$. More formally, we have bijections $f : \mathbb{N}_n \to X - A$ and $g : \mathbb{Z}^+ \to A$, so we can form a bijection $h : \mathbb{Z}^+ \to X$ by letting $h(i) = f(i)$ for $i \leq n$ and $h(i) = g(i - n)$ for $i > n$.)

4. Three people decide to get tacos, and the tacqueria serves five kinds of tacos: beef, chicken, pork, fish, and vegetarian. Each person orders exactly one taco.

   (a) 5 points How many choices are possible if we record who selected which dish (as the waiter should)?

   Solution: Each person can choose one of five types of taco, so there are $5 \cdot 5 \cdot 5 = 5^3 = 125$ possible choices for all three.

   (b) 5 points How many choices are possible if we forget who ordered which dish (as the chef might)?
   Be careful, this is more complicated than it may seem at first.

   Solution: Here there is a slight complication since more than one person might order the same type of taco. We just count the three cases separately.

   - First, if all three get the same type of taco, there are 5 possibilities.
   - If two get the same type of taco, and one gets something else, we have 5 choices for the two that are the same, and 4 choices remain for the different one. This gives us 20 possibilities.
   - Finally, if all three get different types, this means we have $\binom{5}{3} = 10$ possibilities.

   Altogether, this gives us $5 + 20 + 10 = 35$ different orders from the chef's point of view.

5. (a) 7 points Let X and Y be finite sets of the same cardinality. Prove that any surjective function $f : X \to Y$ is also an injection.

   Solution: Suppose $f : X \to Y$ is a surjection, and for contradiction, suppose also that f is not an injection. Then there are x_1 and $x_2 \in X$ so that $f(x_1) = f(x_2)$. This means that the restriction of f to $X - \{x_1\}$ is a surjection onto Y. But since $|X - \{x_1\}| < |Y|$, f cannot be a surjection, giving a contradiction.

   (b) 3 points Suppose that X and Y are infinite sets of the same cardinality. Is it still true that any surjective function $f : X \to Y$ is also injective? Prove or give a counterexample.

   Solution: No, this does not hold. For example, consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x(x-1)(x+1)$. This is a surjective function ($f(x) = y$ has a solution for every choice of y), but it is not injective (since $f(0) = f(1) = f(-1) = 0$).

6. 5 points What is the coefficent of x^9 in the expansion of $(x+2)^{12}$?

Solution: We apply the binomial theorem, which tells us that the term involving x^9 looks like

$$\binom{12}{9} x^9 2^3 = 8\frac{12!}{9!3!}x^9 = 8 \cdot 220x^9 = 1760x^9$$

so the coefficient of x^9 is 1760.