1. (a) 4 points Write a statement that is logically equivalent to the one below, but uses no negatives.

*If you didn’t do the homework, then you won’t pass the exam.*

**Solution:** This is an implication of the form \( \neg P \implies \neg Q \), where the statement \( P \) is “You did the homework”, and the statement \( Q \) is “you pass the exam”. The contrapositive (which is always an equivalent statement) of \( \neg P \implies \neg Q \) is \( Q \implies P \), that is, 

*If you pass the exam, then you did the homework.*

Note that the statement “If you did the homework, you will pass the exam.” is not equivalent to the original. Rather, it is the converse.

(b) 4 points Write the negation of the statement below, using no negatives:

*For every positive real number \( \varepsilon \) and for every integer \( x \), there is an integer \( y \) so that*

\[
0 \leq \frac{x}{y} \quad \text{and} \quad \frac{x}{y} < \varepsilon
\]

**Solution:** Some people found this easier to do by first writing the original symbolically, which is

\[
\forall \varepsilon \in \mathbb{R}^+ \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, 0 \leq \frac{x}{y} \text{ and } \frac{x}{y} < \varepsilon
\]

To negate such a statement, we exchange the quantifiers \( \forall \) and \( \exists \) and negate what follows, giving us

\[
\exists \varepsilon \in \mathbb{R}^+ \exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \neg \left(0 \leq \frac{x}{y} \text{ and } \frac{x}{y} < \varepsilon\right)
\]

Now we write the negation of the innermost part. Recall that \( \neg (A \text{ and } B) \) is \( \neg A \) or \( \neg B \), so we have

\[
\exists \varepsilon \in \mathbb{R}^+ \exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \ 0 > \frac{x}{y} \text{ or } \frac{x}{y} \geq \varepsilon
\]

In words, we would say this as

*There is a positive real number \( \varepsilon \) and an integer \( x \), so that for any integer \( y \) we have either*

\[
\frac{x}{y} < 0 \quad \text{or} \quad \frac{x}{y} \geq \varepsilon
\]
2. **8 points** Prove that for any integer \( n \), if \( n^2 \) is odd, then \( n \) is odd.

**Solution:** It is most straightforward to prove the contrapositive, that is, to show that if \( n \) is even, then \( n^2 \) is also even.

If \( n \) is even, then there is an integer \( q \) so that \( n = 2q \). Then \( n^2 = (2q)^2 = 2(2q^2) \). We have shown there is an integer \( m \) (namely, \( m = 2q^2 \)) so that \( n^2 = 2m \), so \( n^2 \) is even, as desired.

3. **6 points** Prove that for any sets \( A \), \( B \), and \( C \), \( (A \cap C) - B = (A - B) \cap C \)

**Solution:** This can be done in several essentially equivalent ways. The simplest is to note that for any sets \( S \) and \( R \), \( S - R = S \cap R^c \) (where \( R^c \) is the complement of \( R \).

Then we have

\[
(A \cap C) - B = (A \cap C) \cap B^c = A \cap B^c \cap C = (A - B) \cap C.
\]

Another way is to take an element of one set and argue that it lies in the other, and vice-versa. Even though it is essentially equivalent, I’ll do that too:

Suppose \( x \in (A \cap C) - B \). This means that \( x \in A \cap C \) and \( x \notin B \). Since \( x \in (A \cap C) \), we have \( x \in A \) and \( x \in C \). Reordering, we have \( x \in A \) and \( x \notin B \) and \( x \in C \). Putting these together gives us \( x \in (A - B) \cap C \), which shows \( (A \cap C) - B \subseteq (A - B) \cap C \). The argument above is completely reversible, so we also know \((A - B) \cap C \subseteq (A \cap C) - B\), giving the desired result.

Many students chose to do this via a truth table with 8 lines:

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<th>( x \in C )</th>
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<th>( x \in (A \cap C) - B )</th>
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Note that values for membership in both sets agree; specifically, both are false except in the third line of the table.

Finally, a Venn diagram would be acceptable, provided that the regions \( A \cap C \) and \( A - B \) are indicated as well as \( (A \cap C) - B \) and \( (A - B) \cap C \) (the last two are of course the same).
4. **8 points** Prove that for any positive integer \( n \), \( 4^n + 5 \) is divisible by 3. You might find induction helpful. Recall that \( 4 = 3 + 1 \).

\[4^n + 5\]

**Solution:** We’ll do this by induction.

For the base case, notice that if \( n = 1 \) we have \( 4^1 + 5 = 9 \), and 9 is divisible by 3.

Now we show that whenever \( 4^k + 5 \) is divisible by 3, we must also have \( 4^{k+1} + 5 \) divisible by 3. To see this, notice that

\[4^{k+1} + 5 = (3 + 1) \cdot 4^k + 5 = 3 \cdot 4^k + (4^k + 5)\]

Since \( 4^k + 5 \) is divisible by 3 by our inductive hypothesis, there is some integer \( q \) so that \( 4^k + 5 = 3q \). This means we have shown

\[4^{k+1} + 5 = 3 \cdot 4^k + 3q = 3(4^k + q),\]

giving the desired conclusion.

5. Indicate whether each of the following statements is true or false, and justify your answer with a proof.

(a) **3 points** \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y > 0 \)

\[\text{True} \quad \text{False}\]

**Solution:** True. We must show that for any given real number \( x \), we can find a \( y \) so that \( x + y \) is positive. Choosing \( y = 1 - x \) works fine, since \( x + (1 - x) = 1 \) and \( 1 > 0 \). Of course, there are plenty of other choices that work just as well.

(b) **3 points** \( \exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x + y > 0 \)

\[\text{True} \quad \text{False}\]

**Solution:** False. Suppose there were such a value of \( y \); let’s call it \( Q \). Then it would be true that for any choice of \( x \in \mathbb{R} \), \( x + Q \) is positive. If we take \( x = -Q \), this fails. So no such \( Q \) can exist.

An alternative is to prove the negation of the statement is true. That is, we can show that \( \forall y \in \mathbb{R}, \exists x \in \mathbb{R}, x + y \leq 0 \). But this is almost the same as the answer to the previous part: given any such \( y \), let \( x = -1 - y \), and then \( x + y = -1 \). Since the negation of the statement is true, the original statement must be false.

(c) **3 points** \( \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, xy \geq 0 \)

\[\text{True} \quad \text{False}\]

**Solution:** True. Note that if \( x = 0 \), then no matter what \( y \) is, we have \( xy = 0 \cdot y = 0 \), as desired.
6. Let \( f: \mathbb{R} \to \mathbb{R}^2 \) be given by \( f(x) = (x + 1, x^2 + 1) \).

(a) 4 points Is \( f \) surjective? Prove or disprove your answer.

**Solution:** \( f \) is not surjective.

If it were, then for any ordered pair \((a, b) \in \mathbb{R}^2\), we could find \( x \) so that \( f(x) = (a, b) \). But there is no \( x \) so that \( f(x) = (1, 0) \). If there were, then since \( x + 1 = 1 \), we’d have \( x = 0 \). But \( f(0) = (1, 1) \neq (1, 0) \).

(b) 4 points Is \( f \) injective? Prove or disprove your answer.

**Solution:** Yes, \( f \) is injective.

To see this, suppose \( f(x) = f(y) \). Then we have \( (x + 1, x^2 + 1) = (y + 1, y^2 + 1) \), and in particular, \( x + 1 = y + 1 \). This means \( x = y \), and so \( f \) is an injection.