MATH 141 Solutions to Midterm 2

$\boxed{\text{5 pts.}}$ 1. (a) Give a complete and careful definition of the derivative of a function $f(x)$ at the point $x = a$.

Solution:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists.
You could also give the equivalent limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$\boxed{\text{5 pts.}}$ (b) Give a complete statement of the Extreme Value Theorem.

Solution: If f is continuous on a closed interval $[a, b]$, then it attains its maximum and minumum values there. Specifically, there are numbers t and s in $[a, b]$ so that for all $a \leq x \leq b$, we have $f(t) \geq f(x)$ and $f(s) \geq f(x)$.

$\boxed{\text{5 pts.}}$ (c) Let $f : A \to B$ where A and B are both sets of real numbers. Define what the statement "The function g is the inverse of f" means.

Solution: Let $C \subseteq B$ be the image of $f(A)$. Then g is the inverse of f if

- for every $x \in A$ we must have $g(f(x)) = x$, and
- for every $y \in C$, we have $f(g(y)) = y$.

Both conditions are necessary.

$\boxed{\text{5 pts.}}$ (d) Give a definition of the following statement: "The function $f(x)$ has an essential discontinuity at $x = a$."

Solution: $f(x)$ has an essential discontinuity at $x = a$ if

- $f(x)$ is not continuous at $x = a$, and
- there is no number L such that $\lim_{x \to a} f(x) = L$, and consequently there is no way to redefine $f(a)$ so that $f(x)$ is continuous at $x = a$.

2. For each of the functions below, calculate its derivative.

5 pts. (a) $f(x) = \cos 2x \ln 3x$

Solution: Use the product rule and the chain rule to obtain $f'(x) = \dfrac{\cos 2x}{x} - 2\sin 2x \ln 3x$.

5 pts. (b) $g(x) = |x|^3$

Solution: If $x > 0$, we have $g(x) = x^3$ and so $g'(x) = 3x^2$. If $x < 0$, we have $g(x) = -x^3$ and so $g'(x) = -3x^2$. Note that

$$\lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{|h^3|}{h} = \lim_{h\to 0} h|h| = 0,$$

which agrees with both formula for g' at $x = 0$, so $g'(0)$ makes sense and is zero. Thus,

$$g'(x) = \begin{cases} 3x^2 & x \geq 0 \\ -3x^2 & x < 0 \end{cases} = \begin{cases} \frac{3|x|^3}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

5 pts. (c) $h(x) = (1 + \sin x)^x$

Solution: We use logarithmic differentiation here. We have

$$\ln(h(x)) = \ln\left((1 + \sin x)^x\right) = x \ln(1 + \sin x).$$

Thus,

$$\frac{h'(x)}{h(x)} = \ln(1 + \sin x) + x\frac{\cos x}{1 + \sin x}$$

and so

$$h'(x) = (1 + \sin x)^x \left(\ln(1 + \sin x) + x\frac{\cos x}{1 + \sin x}\right).$$

5 pts. (d) $a(x) = \arcsin(2x^{1/2})$

Solution:

$$a'(x) = \frac{1}{\sqrt{1 - (2x^{1/2})^2}} \cdot x^{-1/2} = \frac{1}{\sqrt{x}\sqrt{1 - 4x}} = \frac{1}{\sqrt{x - 4x^2}}.$$

20 pts. 3. Let

$$f(x) = \begin{cases} \sin x & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}.$$

Is $f(x)$ differentiable at $x = 0$? If your answer is yes, calculate $f'(0)$. In either case, justify your answer fully.
Hint: don't panic. Use the force (or maybe problem (1a)).

Solution: Let's use the definition of the derivative

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h}$$

Now let $\{h_k\}$ be any sequence with $h_k \to 0$. Form a subsequence $\{r_i\}$ consisiting of all the rational values of h_k, and $\{q_j\}$ consisting of all the remaining irrational values. Then

$$\lim \frac{f(q_j)}{q_j} = \lim \frac{q_j}{q_j} = 1, \quad \text{and} \quad \lim \frac{f(r_i)}{r_i} = \lim \frac{\sin(r_i)}{r_i} = 1$$

(since $\sin(x)/x \to 1$ and $x \to 0$).
Since $\lim f(q_j)/q_j = \lim f(r_i)/r_i = 1$, we have $\lim(f(h_k)/h_k) = 1$, and consequently, $f'(0) = 1$.

If you don't want to use sequences, you could just observe that for $h \neq 0$, we have $\sin(h)/h \leq f(h)/h \leq 1$ and use the squeeze theorem. It is really the same thing.

20 pts. 4. The Fundamental Theorem of Algebra tells us that if $p_n(z)$ is any polynomial of degree n, it can factored into a constant times the product of n linear terms. That is, there are complex numbers $\rho_j, \quad j = 1 \dots n$ and a constant a so that

$$p_n(z) = a(z - \rho_1)(z - \rho_2)(z - \rho_3) \cdots (z - \rho_n).$$

Use this to show (using induction on the degree n) that the derivative of any polynomial p_n can be written in the form

$$p_n'(z) = p_n(z) \sum_{j=1}^{n} \frac{1}{z - \rho_j}, \qquad (*)$$

provided $p_n(z) \neq 0$.

Hint: you may assume $a = 1$ to make things a little easier, since if $a \neq 1$ we may instead work with the polynomial $q(z) = p(z)/a$.

Solution: As mentioned in the hint, we may assume that p is a monic polynomial, that is, $a = 1$.

For the base case, we observe that if $p_1(z) = z + b$, we have $p_1'(z) = 1 = \frac{p_1(z)}{z+b}$ (here $\rho_1 = -b$).

For the inductive step, we want to show that as long as we know that every polynomial p_{n-1} of degree $n-1$ satisfies (*), we also know it holds for any polynomial p_n of degree n.

So, let $p_n(z)$ be any monic degree n polynomial. By the fundamental theorem of algebra,

$$p_n(z) = (z - \rho_1)(z - \rho_2)(z - \rho_3) \cdots (z - \rho_n).$$

Now let $p_{n-1}(z) = (z - \rho_1)(z - \rho_2)(z - \rho_3) \cdots (z - \rho_{n-1})$, and so $p_n(z) = p_{n-1}(z) \cdot (z - \rho_n)$, and $p_{n-1}(z) = p_n(z)/(z - \rho_n)$ (as long as $z \neq \rho_n$).

Using the product rule, we have $p_n'(z) = p_{n-1}'(z) \cdot (z - \rho_n) + p_{n-1}(z)$, and then, since the degree of $p_{n-1}(z)$ is $n-1$, we apply the inductive hypothesis to get

$$\begin{aligned}
p_n'(z) &= p_{n-1}(z)\Big(\sum_{j=1}^{n-1} \frac{1}{z - \rho_j}\Big) \cdot (z - \rho_n) + p_{n-1}(z) \\
&= p_n(z)\Big(\sum_{j=1}^{n-1} \frac{1}{z - \rho_j}\Big) + \frac{p_n(z)}{z - \rho_n} \\
&= p_n(z)\Big(\sum_{j=1}^{n} \frac{1}{z - \rho_j}\Big),
\end{aligned}$$

as desired.

Alternatively, you could use logarithmic differentiation, to turn the product into a sum of logs.