MATH 141 Solutions to Midterm 2

5 pts. 1. (a) Give a complete and careful definition of the derivative of a function $f(x)$ at the point $x = a$.

Solution:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists.
You could also give the equivalent limit

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

5 pts. (b) Give a complete statement of the Mean Value Theorem.

Solution: Suppose that f is differentiable on a nontrivial interval (a, b), and continuous on $[a, b]$. Then there is at least one point $c \in (a, b)$ so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

5 pts. (c) Let $f : A \to B$ where A and B are both sets of real numbers. Define what the statement "The function g is the inverse of f" means.

Solution: Let $C \subseteq B$ be the image of $f(A)$. Then g is the inverse of f if

- for every $x \in A$ we must have $g(f(x)) = x$, and
- for every $y \in C$, we have $f(g(y)) = y$.

Both conditions are necessary.

5 pts. (d) Give a definition of the following statement: "The function $f(x)$ has a removable discontinuity at $x = a$."

Solution: $f(x)$ has a removable discontinuity at $x = a$ if

- $f(x)$ is not continuous at $x = a$, but
- there is a number L such that $\lim_{x\to a} f(x) = L$, and consequently the function $g(x) = \begin{cases} f(x) & \text{if } x \neq a \\ L & \text{if } x = a \end{cases}$ is continuous at $x = a$.

20 pts. 2. For every integer $n \geq 2$, suppose that $g_n(x)$ is the product of n differentiable functions $f_1(x), f_2(x), \ldots, f_n(x)$. Prove that if $g_n(x) \neq 0$, then

$$g_n'(x) = g_n(x) \sum_{k=1}^{n} \frac{f_k'(x)}{f_k(x)}$$

You might find induction helpful.

Solution: For typographical simplicity, I will write g_n instead of $g_n(x)$, f_k instead of $f_k(x)$, etc.

We establish the result by induction on n. Observe that since $g_n \neq 0$, we must have $f_k \neq 0$ for all k.

First, the base case ($n = 2$) is the ordinary product rule. That is, $g_2 = f_1 f_2$ and so

$$g_2' = f_1' f_2 + f_1 f_2' = f_1 f_2 \left(\frac{f_1'}{f_1} + \frac{f_2'}{f_2} \right) = g_2 \sum_{k=1}^{2} \frac{f_k'}{f_k}.$$

For the inductive step, we must show that if the result holds for g_{n-1}, it also holds for g_n. Thus we have

$$\begin{aligned}
g_n' = (f_n g_{n-1})' &= f_n' g_{n-1} + f_n g_{n-1}' && \text{by applying the product rule} \\
&= f_n' g_{n-1} + f_n \left(g_{n-1} \sum_{k=1}^{n-1} \frac{f_k'}{f_k} \right) && \text{applying the inductive hypothesis} \\
&= f_n' \frac{f_n}{f_n} g_{n-1} + f_n g_{n-1} \sum_{k=1}^{n-1} \frac{f_k'}{f_k} \\
&= \frac{f_n'}{f_n} g_n + g_n \sum_{k=1}^{n-1} \frac{f_k'}{f_k} && \text{since } f_n g_{n-1} = g_n \\
&= g_n \left(\frac{f_n'}{f_n} + \sum_{k=1}^{n-1} \frac{f_k'}{f_k} \right) \\
&= g_n \sum_{k=1}^{n} \frac{f_k'}{f_k} && \text{as desired.}
\end{aligned}$$

Since we established the fact for $n = 2$ and showed that if the result holds for $n - 1$ it must also hold for n, we have established it for every integer greater than 1.

Note that this multi-term product rule is probably easier to see as

$$(f_1 f_2 f_3 \cdots f_n)' = f_1' f_2 f_3 \cdots f_n \; + \; f_1 f_2' f_3 \cdots f_n \; + \; f_1 f_2 f_3' \cdots f_n \; + \ldots + \; f_1 f_2 f_3 \cdots f_n'.$$

20 pts. 3. Suppose that f is a continuous function from the interval $[0, 1]$ with the property that $0 \leq f(x) \leq 1$ for every $x \in [0, 1]$. Show that there is at least one number $c \in [0, 1]$ such that $f(c) = c$. (Hint: consider the function $g(x) = f(x) - x$.)

Solution: Observe that if $f(0) = 0$ or $f(1) = 1$, we have found such a c. So let us assume that $f(0) \neq 0$ and $f(1) \neq 1$. Since $0 \leq f(x) \leq 1$, we must have $f(0) > 0$ and $f(1) < 1$.

Now, let $g(x) = f(x) - x$. Because f is continuous on $[0, 1]$, g is also continuous. Since $f(0) > 0$, we know $g(0) = f(0) > 0$. Also, since $f(1) < 1$, we have $g(1) = f(1) - 1 < 0$. We now apply the intermediate value theorem: Since g is continuous on $[0, 1]$ with $g(0) < 0$ and $g(1) > 0$, there must be a value $c \in [0, 1]$ with $g(c) = 0$. But $g(c) = f(c) - c$, and so $f(c) = c$, as desired.

20 pts. 4. Let

$$f(x) = \begin{cases} |x|^x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Show that $f(x)$ is a continuous function for all $x \in \mathbb{R}$.

Solution: First, note that for $x \neq 0$, we know $f(x)$ is continuous: for $x > 0$, we have $|x| = x$, and so

$$f(x) = x^x = e^{\ln x^x} = e^{x \ln x}$$

Since this is a composition of continuous functions, it is continuous. Similarly, for $x < 0$, we have

$$f(x) = (-x)^x = e^{x \ln(-x)},$$

again a composition of continuous functions. (Note that $\ln(-x)$ is well defined, because $x < 0$ so $-x > 0$.)

This means the only concern is at $x = 0$. We need to confirm that $\lim_{x \to 0} |x|^x = 1$. Use L'Hôpital's rule and happiess ensues.

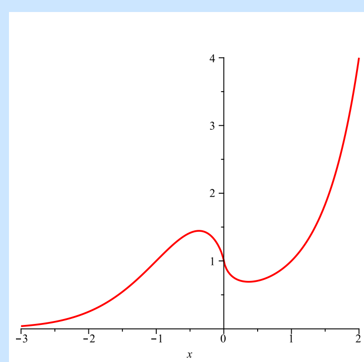
More specifically, since $|x|^x = e^{x \ln |x|}$, let's look at the behaviour of $x \ln |x|$ as $x \to 0$. We use L'Hôpital's rule.

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} -x = 0$$

A nearly identical computation shows $\lim_{x \to 0^-} x \ln |x| = 0$. Thus

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} |x|^x = e^{\lim x \ln |x|} = e^0 = 1 = f(0)$$

so f is continuous at $x = 0$.



The graph of $y = x^x$

5. For each of the functions below, calculate its derivative.

5 pts. (a) $e^{2x}\sec 3x$

Solution:

$$2e^{2x}+3e^{2x}\sec 3x\tan 3x$$

5 pts. (b) $\arctan\sqrt{x-1}$

Solution:

$$\frac{1}{1+(x-1)}\cdot\frac{1}{2\sqrt{x-1}}=\frac{1}{2x\sqrt{x-1}}$$

5 pts. (c) $\dfrac{(x^5+2x^3+8)\sqrt{x+x^2}}{(\sin 2x+\cos 2x)(x^4-x^2+5)}$

Solution: We use logarithmic differentiation here. Let $y=\frac{(x^5+2x^3+8)\sqrt{x+x^2}}{(\sin 2x+\cos 2x)(x^4-x^2+5)}$, so

$$\ln y=\ln(x^5+2x^3+8)+\frac{1}{2}\ln(x+x^2)-\ln(\sin 2x+\cos 2x)-\ln(x^4-x^2+5).$$

Thus,

$$\frac{y'}{y}=\frac{x^4+6x^2}{x^5+2x^3+8}+\frac{1+2x}{2(x+x^2)}-\frac{2\cos 2x-2\sin 2x}{\sin 2x+\cos 2x}-\frac{4x^3-2x}{x^4-x^2+5},$$

and so

$$y'=\frac{(x^5+2x^3+8)\sqrt{x+x^2}}{(\sin 2x+\cos 2x)(x^4-x^2+5)}\left(\frac{x^4+6x^2}{x^5+2x^3+8}+\frac{1+2x}{2(x+x^2)}-\frac{2\cos 2x-2\sin 2x}{\sin 2x+\cos 2x}-\frac{4x^3-2x}{x^4-x^2+5}\right).$$

(what a mess!)

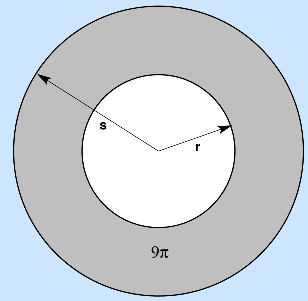
5 pts. (d) $\ln(\ln(\ln(x)))$

Solution:

$$\frac{1}{\ln(\ln(x))}\cdot\frac{1}{\ln(x)}\cdot\frac{1}{x}$$

20 pts. 6. The area between two varying concentric circles is 9π cm^2 at all times. The rate of change of the area of the larger circle is 10π cm^2/sec. How fast is the circumference of the smaller circle changing when its area is 16π cm^2?

Solution: Let r be the radius of the smaller circle, and s be the radius of the larger circle. We know that $\pi s^2 - \pi r^2 = 9\pi$, and that the rate of change of the area of the larger circle is 10π. This means $\frac{d}{dt}\pi s^2 = 10\pi$, so $2\pi s\frac{ds}{dt} = 10\pi$, and so $s\frac{ds}{dt} = 5$.



We want to find the rate of change of the circumference of the smaller circle. Since this circumference is $C = 2\pi r$, we have $\frac{dC}{dt} = 2\pi\frac{dr}{dt}$. So finding $\frac{dr}{dt}$ is sufficient.

Since $\pi s^2 - \pi r^2 = 9\pi$, differentiating both sides with respect to t gives

$$2s\frac{ds}{dt} = 2r\frac{dr}{dt}.$$

When the area of the small circle is 16π, its radius r is 4. We also know from above that $s\frac{ds}{dt} = 5$.

It doesn't actually matter, but the radius of the larger circle at this time must be 5 (since its area is $16\pi + 9\pi = 25\pi$), and since $s\frac{ds}{dt} = 5$, we know $\frac{ds}{dt} = 1$.

Plugging these into the above relationship gives

$$2 \cdot 5 = 2 \cdot 4 \cdot \frac{dr}{dt} \qquad \text{so} \qquad \frac{5}{4} = \frac{dr}{dt}.$$

Thus, the rate of change of the circumference of the small circle is $\frac{5\pi}{2}\ \frac{\text{cm}}{\text{sec}}$.