EXAM

Final Exam

Math 131

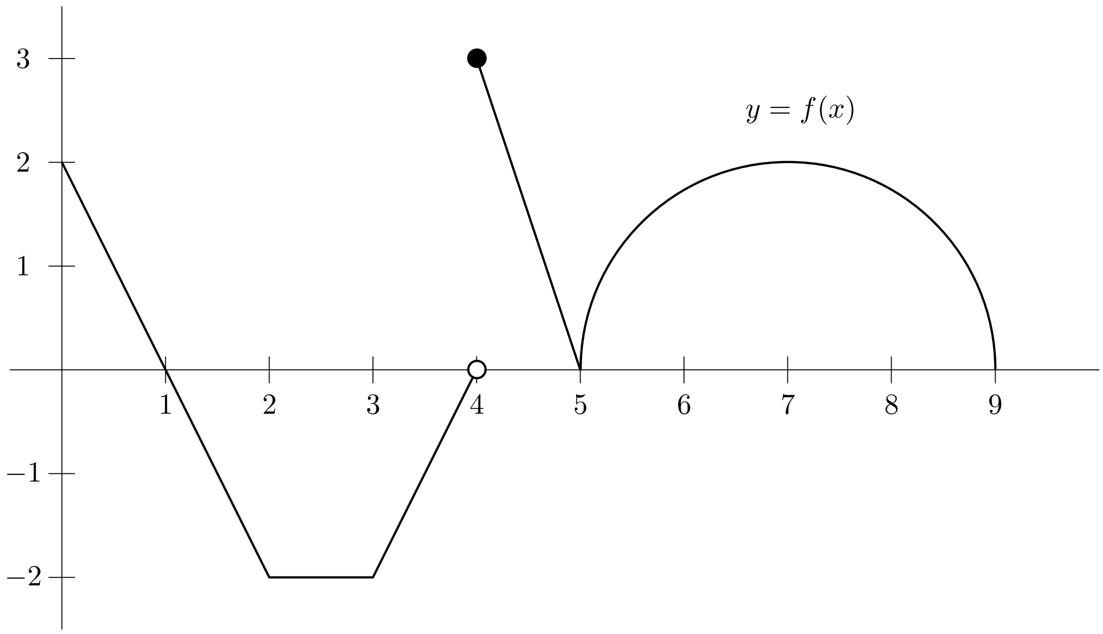
Tuesday, December 16, 2003

ANSWERS

Problem 1. **[12 points]** Let $h(x) = \dfrac{1}{x^2+1}$. On what intervals is the graph of h concave up?

Answer:
We compute
$$h'(x) = \frac{-2x}{(x^2+1)^2}$$
and
$$h''(x) = \frac{(x^2+1)^2(-2) - (-2x)(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{-2(x^2+1) + (8x^2)}{(x^2+1)^3} = \frac{6x^2-2}{(x^2+1)^3}.$$
The graph of h is concave up when
$$h''(x) = \frac{6x^2-2}{(x^2+1)^3} > 0 \Leftrightarrow 6x^2 - 2 > 0 \Leftrightarrow x^2 > \frac{1}{3}.$$
Thus, h is concave on the intervals $\left(-\infty, \frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \infty\right)$.

Problem 2. **[3 points each]** Use the sketch of $y = f(x)$ below to find the following:



Answer:

(a) $f'(1) = -2$

(b) $\int_0^3 f(x)dx = -2$

(c) $f(4) = 3$

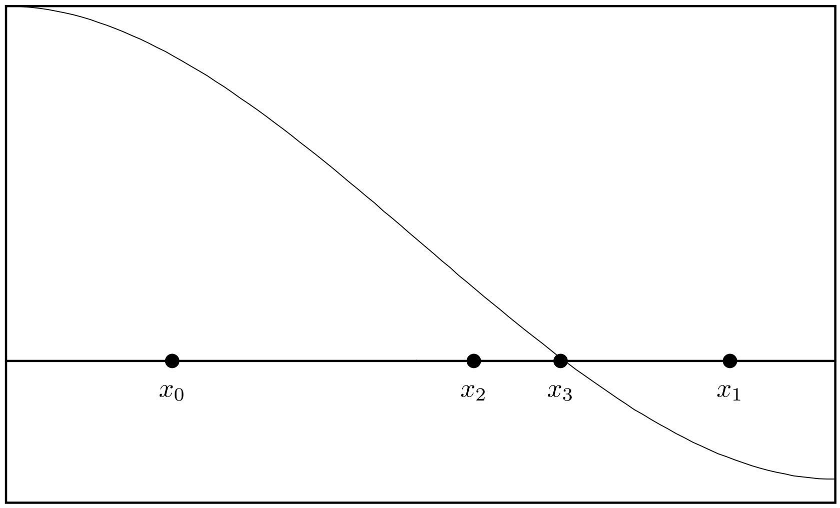
(d) $\lim_{x \to 4^-} f(x) = 0$

(e) $\int_5^9 f(x)dx = 2\pi$

(f) $f'(5) =$ does not exist.

Problem 3. Below is the graph of a function g. Consider Newton's method for solving the equation $g(x) = 0$.

Answer:



(a) **[6 points]** The four points in the figure above represent an initial guess and three values produced from successive iterations of Newton's method. But which are which? Correctly label the points in the picture: mark the initial guess as x_0 and the other points as x_1, x_2, and x_3.

(b) **[6 points]** Here, actually,

$$x_0 = \frac{2}{5} \quad g\left(\frac{2}{5}\right) = \frac{43}{50} \text{ and } g'\left(\frac{2}{5}\right) = -\frac{16}{25}.$$

Use this information to calculate the value of x_1.

$$x_1 = x_0 - \frac{g(x_0)}{g'(x_0)} = \frac{2}{5} - \frac{\frac{43}{50}}{-\frac{16}{25}} = \frac{2}{5} + \frac{43}{32} = \frac{279}{160} = 1.74375.$$

Problem 4. [**3 points each**] True or False.

Answer:

(a) Suppose that f is differentiable and $f(2) = f(6)$. Then there must be at least one point $c \in (2, 6)$ with $f'(c) = 0$. **True** by the mean value theorem.

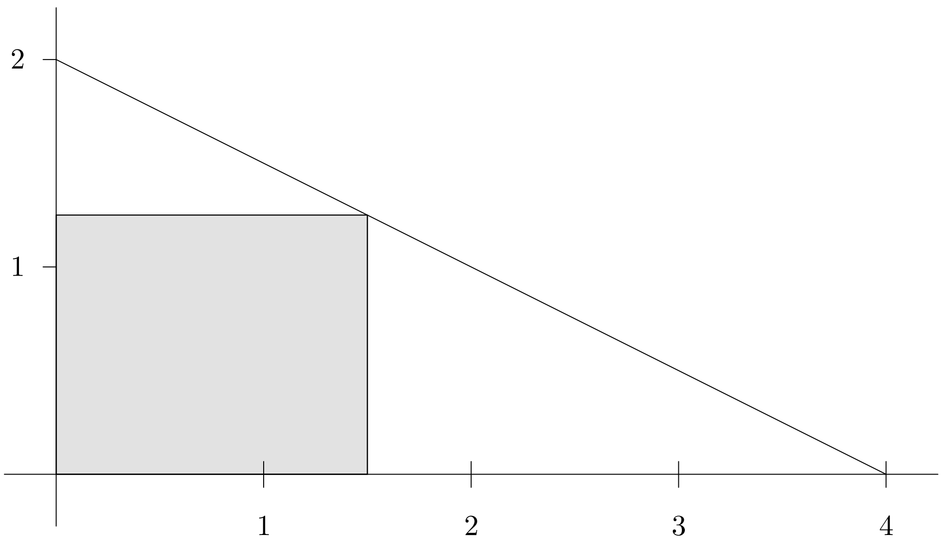
(b) Suppose that g is continuous, $g(1) = 5$ and $g(5) = 10$. Then the equation $g(c) = 7$ must have a solution $c \in (1, 5)$. **True** by the intermediate value theorem.

(c) If $f(x) > x$ for all x, then $\displaystyle\int_0^{10} f(x)dx > 5$. **True**: since $5 = \int_0^1 0xdx$.

(d) If $f'(x) = g'(x)$ then $f(x) = g(x)$. **False.**

(e) $\displaystyle\int_a^b f(x)g(x)dx = \left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right)$ for any f and g. **False.**

Problem 5. **[12 points]** Pictured below is the triangle formed by the intersection of the line $y = -\frac{1}{2}x + 2$, the x-axis, and the y-axis, together with an inscribed rectangle.



Find the area of the largest rectangle that can be inscribed in this triangle.

Answer:
We wish to maximize $A = xy$ subject to the constraint that $y = -\frac{1}{2}x+2$, and $x \in [0, 4]$. Substituting $-\frac{1}{2}x + 2$ in for y gives

$$A(x) = -\frac{1}{2}x^2 + 2x.$$

In order to maximize A, we find

$$A'(x) = -x + 2.$$

So the critical points for A are $x = 0, 2, 4$. We note that A is continuous on $[0, 4]$ and thus will have an absolute maximum among $A(0) = 0, A(2) = 2, A(4) = 0$. So, the largest rectangle that can be inscribed in this triangle has dimensions $x = 2$, $y = 1$, and has an area of 2.

Problem 6. **[4 points each]** Compute:

Answer:

(a) $\lim_{x\to 0} \dfrac{x\sin(x)}{\cos(x)-1} \stackrel{L}{=} \lim_{x\to 0} \dfrac{x\cos(x)+\sin(x)}{-\sin(x)} \stackrel{L}{=} \lim_{x\to 0} \dfrac{\cos(x)-x\sin(x)+\cos(x)}{-cos(x)} = -2.$

(b) $\displaystyle\int_0^{\frac{\pi}{3}} \sin(x)\,dx = \cos(x)\Big]_0^{\frac{\pi}{3}} = 1 - \frac{1}{2} = \frac{1}{2}.$

(c) $\dfrac{d}{dx}\sin\left(e^x\right) = e^x\cos(e^x).$

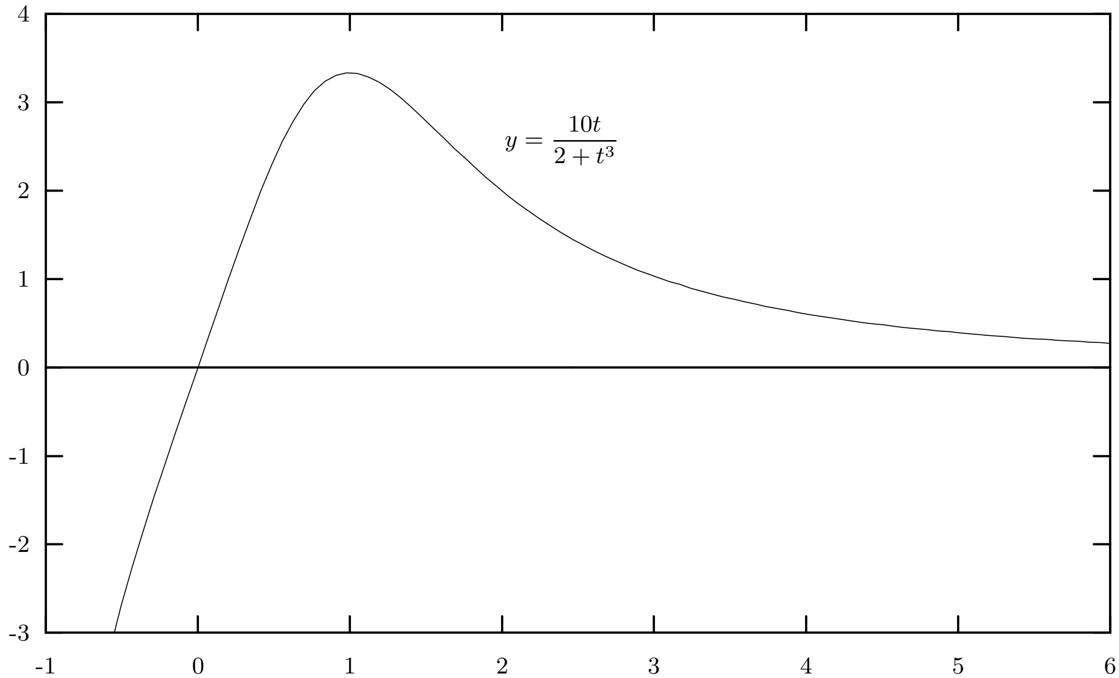
(d) $\displaystyle\int x^2+3x+5dx = \frac{x^3}{3} + \frac{3x^2}{2} + 5x + c.$

(e) $\lim_{x\to\infty} \dfrac{3x-e^x}{8x^2+\ln(x)} = -\infty.$

Problem 7. Let A be the function defined by

$$A(x) = \int_1^x \frac{10t}{2+t^3} dt \text{ for } t \geq -\sqrt[3]{2}.$$

This sketch might be helpful:



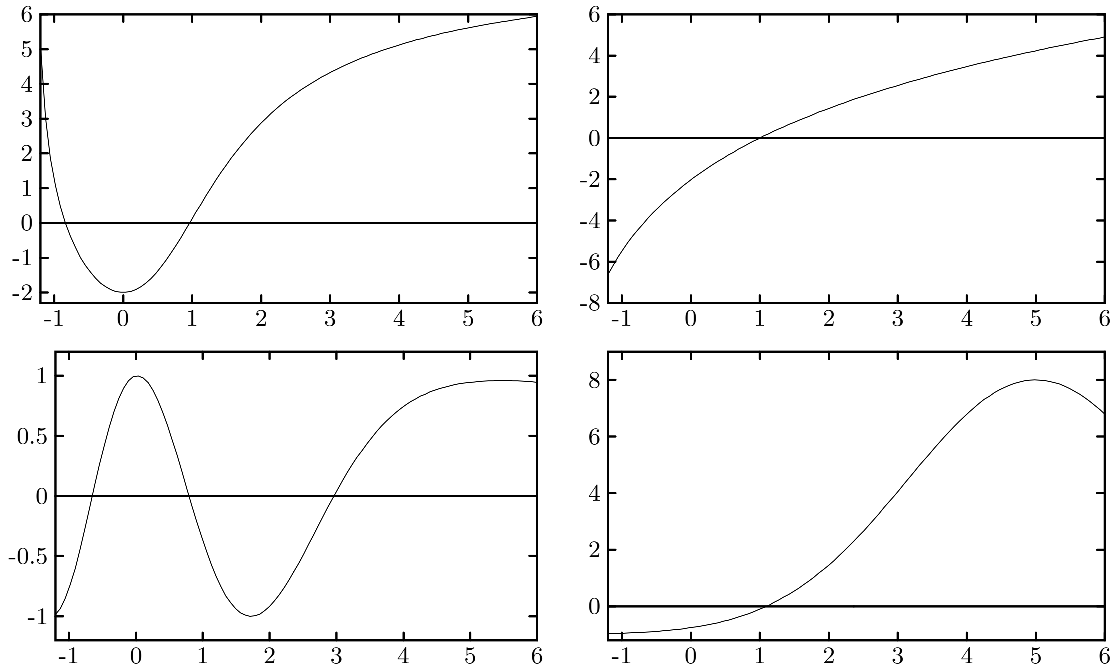
(a) **[6 points]** Approximate $A(3)$ using two circumscribed rectangles.

Answer:

$$A(3) \approx (height_1)(width_1) + (height_2)(width_2) = \left(\frac{10}{3}\right)(1) + \left(\frac{20}{10}\right)(1) = \frac{16}{3} = 5.33333\ldots.$$

Problem 7. (Continued.)

(b) **[6 points]** Which of the following is the graph of $y = A(x)$?



Answer:

The graph of A is the first one. Notice that $A'(x) = \frac{10x}{2+x^3}$. Hence, A is increasing for $x > 0$ and decreasing for $x < 0$. This makes our choice conclusive.