MAT 141 Honors Calculus
Exam 1
SOLUTIONS

1. (3 points each)

   (a) Suppose that A and B are sets. Give the definitions of $A \cup B$ and $A \cap B$.

   $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$; this is the union of A and B
   $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$; this is the intersection of A and B

   (b) What does it mean for an interval to be bounded?

   It means there exist $a, b \in \mathbb{R}$ such that $a \leq x \leq b$ for all x in the interval.

   (c) What does it mean for a sequence to be bounded?

   If the sequence is written $\{a_n\}_{n=0}^{\infty}$, it means there exists some $M \in \mathbb{R}$ such that $|a_n| \leq M$ for all n.

   (d) Give the precise meaning of the phrase, "the sequence $\{a_n\}_{n=0}^{\infty}$ converges to L."

   Given any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ whenever $n \geq N$.

   (e) State the Completeness Axiom.

   Every bounded, monotone sequence converges.

   (f) State the Bolzano–Weierstrass Theorem.

   Every bounded sequence has a convergent subsequence.

   (g) State the Comparison Test for positive series.

   If $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ are series such that $0 \leq a_k \leq b_k$ for all k and $\sum_{k=0}^{\infty} b_k$ converges, then $\sum_{k=0}^{\infty} a_k$ also converges.

   (h) State the Alternating Series Test.

   If $\{a_k\}$ is a strictly decreasing sequence of positive numbers such that $a_k \to 0$ as $k \to \infty$, then the series $\sum_{k=0}^{\infty} (-1)^k a_k$ converges.

2. For each of the following sequences and series, find its limit or state that it diverges. (3 points each)

   (a) $(-1)^n + \dfrac{1}{n}$

   This sequence *diverges;* it has two distinct accumulation points, 1 and -1.

   (b) $\dfrac{3n^3 - 1}{2 - n^2 + n^3}$

   This sequence converges to $\boxed{3}$; it is a rational function of n with equal degree in the numerator and denominator.

   (c) $\dfrac{\cos k}{k!}$

   This sequence converges to $\boxed{0}$, by the Squeeze Theorem, for example.

   (d) $1 - \dfrac{\pi^2}{2} + \dfrac{\pi^4}{24} - \dfrac{\pi^6}{720} + \cdots + (-1)^k \dfrac{\pi^{2k}}{(2k)!} + \cdots$

   This series converges to $\boxed{\cos \pi = -1}$; here we simply recognize the form of the series expression for the cosine.

   (e) $\displaystyle\sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)\left(-\frac{1}{4}\right)^k$

   This is a geometric series with ratio $-1/4$, whose absolute value is less than 1. It converges to

$$\frac{-1/2}{1 - (-1/4)} = \boxed{-\frac{2}{5}}$$

   (f) $\displaystyle\sum_{k=1}^{\infty} \frac{3}{2k}$

   This series *diverges* by comparison with the harmonic series: $\frac{3}{2k} > \frac{1}{k}$ for all k.

3. Use induction to prove that

$$3 + 11 + \cdots + (8n - 5) = 4n^2 - n$$

for all integers $n \geq 1$. (15 points)

First, we establish that the equality is true for $n = 1$. On the left we have simply 3, and on the right we have $4(1)^2 - 1 = 3$. Thus the equality holds for $n = 1$.

Now assume that the equality holds for some fixed n. We want to show that it is true when n is replaced by $n + 1$. By adding $8(n + 1) - 5$ to both sides of the equality given by n, we obtain

$$3 + 11 + \cdots + (8n - 5) + (8(n + 1) - 5) = 4n^2 - n + (8(n + 1) - 5).$$

The right side becomes $4n^2 - n + 8n + 8 - 5 = 4n^2 + 7n + 3$. On the other hand, if we substitute $n + 1$ directly into the expression $4n^2 - n$, we get $4(n + 1)^2 - (n + 1) = 4(n^2 + 2n + 1) - (n + 1) = 4n^2 + 7n + 3$. Thus the equality is also true for $n + 1$.

Because the equality is true for $n = 1$ and true for $n + 1$ whenever it is true for n, it is true for all $n \geq 1$.

4. Find all accumulation points of the sequence $z_n = i^n$. (10 points)

The sequence z_n is periodic:

$$1, i, -1, -i, 1, i, -1, -i, \ldots,$$

with $z_{4n+k} = z_k$ for all $n, k \in \mathbb{N}$. Thus the four values $\{1, -1, i, -i\}$ all appear infinitely often, and because they are the only values that appear, they are the accumulation points of the sequence.

5. Suppose that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are convergent sequences and that $b_n - a_n$ converges to 0. Show that a_n and b_n have the same limit. (18 points)

Set $A = \lim_{n\to\infty} a_n$ and $B = \lim_{n\to\infty} b_n$. We want to show $A = B$. We will give three proofs of this result, all related.

(1) This is the most direct proof, in the sense that it only requires using the definitions, but also the longest.

The equality $A = B$ means the same as $|B - A| < \varepsilon$ for all $\varepsilon > 0$.
So take ε to be any positive number. By the definition of a limit, we know we can choose:

- N_A such that $|a_n - A| < \varepsilon/3$ whenever $n \geq N_A$;
- N_B such that $|b_n - B| < \varepsilon/3$ whenever $n \geq N_B$;
- N_0 such that $|b_n - a_n| < \varepsilon/3$ whenever $n \geq N_0$.

After finding these, set $N = \max\{N_A, N_B, N_0\}$. Then for any $n \geq N$, we have

$$\begin{aligned}
|B - A| &= |B + (-b_n + b_n) + (-a_n + a_n) - A| \\
&= |(B - b_n) + (b_n - a_n) + (a_n - A)| \\
&\leq |B - b_n| + |b_n - a_n| + |a_n - A| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

and thus we have shown that $|B - A| < \varepsilon$. Because ε was arbitrary, we conclude that $A = B$.

(2) This is a variant of the above proof, but it proceeds by contradiction.

Suppose, for the sake of contradiction, that $A \neq B$. Then $|B - A| > 0$, and so $\varepsilon_0 = \frac{|B-A|}{3}$ is also > 0. Therefore there exist N_A, N_B such that $|a_n - A| < \varepsilon_0$ whenever $n \geq N_A$ and $|b_n - B| < \varepsilon_0$ whenever $n \geq N_B$. But this means that, for all $n \geq \max\{N_A, N_B\}$, $|b_n - a_n| > \varepsilon_0$. Formally,

$$\begin{aligned}
|b_n - a_n| &= |b_n - B + B - A + A - a_n| \\
&\geq |B - A| - (|b_n - B| + |a_n - A|) \\
&> |B - A| - 2\frac{|B - A|}{3} = \varepsilon_0.
\end{aligned}$$

This contradicts the assumption that $b_n - a_n \to 0$. Therefore our claim that $A \neq B$ must be false; i.e., we conclude that $A = B$.

(3) This is the "easiest" proof, in the sense that all of the work is hidden in the proof of a theorem we use.

By the theorem on the arithmetic of sequences, we have

$$B = \lim_{n\to\infty} b_n = \lim_{n\to\infty} (b_n - a_n + a_n) = \lim_{n\to\infty} (b_n - a_n) + \lim_{n\to\infty} a_n = 0 + A = A.$$

6. (a) Using the series definition of e^x, show that $e^a > 1$ whenever $a > 0$. (8 points)

Recall that the series definition of e^x is

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

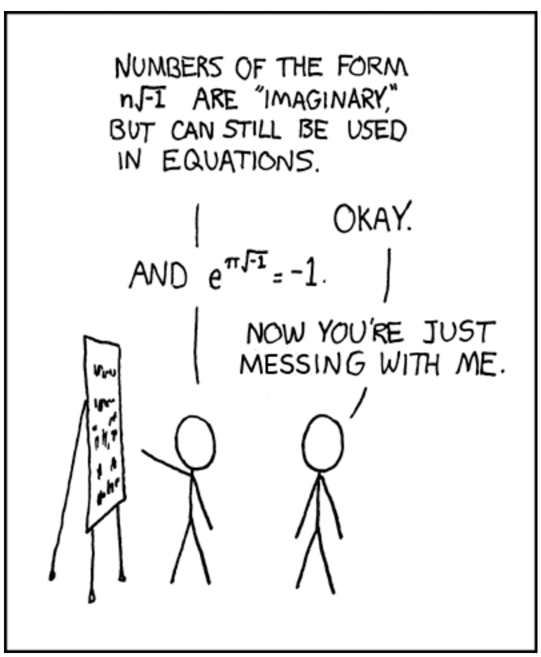
If $a > 0$, then all terms of the series for e^a are strictly positive, and the first term is 1. Therefore the sum of the series is strictly greater than 1.

(b) Show that if $b < 0$, then $0 < e^b < 1$. (*Hint:* Think of b as $-a$ for some positive a and use the key property of exponentials.) (7 points)

Following the hint, we set $a = -b$, so that $a > 0$. Then by the key property of the exponential, we have

$$e^b e^a = e^{b+a} = e^{b-b} = e^0 = 1,$$

which means $e^b = 1/e^a$. Because $e^a > 1$, e^b is positive and less than 1.



comic by Randall Munroe
`http://xkcd.com/179/`