Math 132  Solutions to Midterm 1 (instrumental)

20 pts  1. \[ \int \frac{4x - 5}{x^3 - x} \, dx \]

**Solution:** We do this by partial fractions. Since \( x^3 - x = x(x - 1)(x + 1) \), we want to find numbers \( A, B, \) and \( C \) so that

\[
\frac{4x - 5}{x(x + 1)(x - 1)} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{x + 1}.
\]

Cross-multiplying, we have \( A(x - 1)(x + 1) + B(x + 1) + C(x - 1) = 4x - 5 \).

- If \( x = 0 \), we have \( -A = -5 \) so \( A = 5 \)
- If \( x = 1 \), we have \( 2B = -1 \) so \( B = -1/2 \)
- If \( x = -1 \), we have \( 2C = -9 \) so \( C = -9/2 \).

This means that

\[
\int \frac{4x - 5}{x^3 - x} \, dx = \int \frac{5}{x} - \frac{1/2}{x - 1} - \frac{9/2}{x + 1} \, dx = 5 \ln |x| - \frac{1}{2} \ln |x - 1| - \frac{9}{2} \ln |x + 1| + C.
\]

20 pts  2. \[ \int_4^6 \frac{dr}{(r - 5)^4} \]

**Solution:** Since \( \frac{1}{(r - 5)^4} \) becomes undefined when \( r = 5 \), we have write the integral as the sum of two improper integrals. That is,

\[
\int_4^6 \frac{dr}{(r - 5)^4} = \int_4^5 \frac{dr}{(r - 5)^4} + \int_5^6 \frac{dr}{(r - 5)^4} \\
= \lim_{t \to 5^-} \int_4^t \frac{dr}{(r - 5)^4} + \lim_{t \to 5^+} \int_t^6 \frac{dr}{(r - 5)^4} \\
= \lim_{t \to 5^-} \left[ \frac{-1}{3(r - 5)^3} \right]_4^t + \lim_{t \to 5^+} \left[ \frac{-1}{3(r - 5)^3} \right]_t^6 \\
= \lim_{t \to 5^-} \left( \frac{-1}{3(t - 5)^3} - \frac{1}{3} \right) + \lim_{t \to 5^+} \left( \frac{1}{3(t - 5)^3} \right).
\]

However, both of the limits above are undefined, so the integral diverges.
3. \[ \int_1^2 \frac{\sqrt{t^2 - 1}}{t} \, dt \]

**Solution:** We make the substitution \( t = \sec \theta \), and so \( dt = \sec \theta \tan \theta \, d\theta \). Also, when \( t = 1 \) we have \( \theta = 0 \), and when \( t = 2 \) we have \( \theta = \pi/3 \). Thus, we have

\[
\int_1^2 \frac{\sqrt{t^2 - 1}}{t} \, dt = \int_0^{\pi/3} \frac{\sqrt{\sec^2 \theta - 1}}{\sec \theta} \sec \theta \tan \theta \, d\theta = \int_0^{\pi/3} \sqrt{\tan^2 \theta} \tan \theta \, d\theta
\]

\[
= \left[ \tan \theta - \theta \right]_0^{\pi/3} = \tan(\pi/3) - \frac{\pi}{3} = \sqrt{3} - \frac{\pi}{3}.
\]

4. \[ \int x^5 \cos(x^3) \, dx \]

**Solution:** First, we make the substitution \( w = x^3 \), so \( dw = 3x^2 \, dx \).

\[
\int x^5 \cos(x^3) \, dx = \frac{1}{3} \int w \cos w \, dw = \frac{1}{3} \left( w \sin w - \int \sin w \, dw \right)
\]

\[
= \frac{1}{3} (w \sin w + \cos w) + C = \frac{x^3 \sin x^3 + \cos x^3}{3} + C
\]

where above we used integration by parts, with \( u = w \) and \( dv = \cos w \, dw \), so \( du = dw \) and \( v = \sin w \).

5. \[ \int \cos^3(x) \sin^4(x) \, dx \]

**Solution:** We use the identity \( \cos^2 x + \sin^2 x = 1 \) to transform all but one of the \( \cos x \) terms into powers of \( \sin x \). So we have

\[
\int \cos^3(x) \sin^4(x) \, dx = \int (1 - \sin^2 x) \sin^4 x \cos x \, dx.
\]

Now we make the substitution \( u = \sin x \) with \( du = \cos x \, dx \) to get

\[
\int (1 - u^2)u^4 \, du = \int (u^4 - u^6) \, du = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C.
\]
6. \( \int_0^1 xe^{5x} \, dx \)

**Solution:** We integrate by parts, with \( u = x \) and \( dv = e^{5x} \, dx \), so \( du = dx \) and \( v = \frac{1}{5} e^{5x} \).

This gives us

\[
\int_0^1 xe^{5x} \, dx = \left. \frac{x}{5} e^{5x} \right|_0^1 - \frac{1}{5} \int_0^1 e^{5x} \, dx = \left. \frac{x}{5} e^{5x} - \frac{1}{25} e^{5x} \right|_0^1 = \frac{e^5}{5} - \frac{e^5}{25} + \frac{1}{25}
\]

7. The values of a function \( f(x) \) are given by the table at right.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.5</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1.5</td>
<td>2</td>
</tr>
<tr>
<td>2.5</td>
<td>1</td>
</tr>
</tbody>
</table>

(a) Use Simpson’s rule to approximate \( \int_0^2 f(x) \, dx \).

**Solution:** First, observe that there are two “extra” values in the table, namely \( x = -0.5 \) and \( x = 2.5 \). If we were to include those values (as some people did), the approximation would be for \( \int_{-0.5}^{2.5} f(x) \, dx \).

We have \( \Delta x = 1/2 \), and \( n = 4 \). Simpson’s rule is given by

\[
S_4 = \frac{1}{3} \left( f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + f(2) \right) = \frac{1}{6} (0 + 12 + 4 + 8 - 1) = \frac{23}{6}.
\]

(b) If you know that \(-5x^2 \leq f^{(4)}(x) \leq 5x^2\), what is the maximum error in the approximation above?\(^1\)

**Solution:** This is just a matter of filling in the proper values into the given formula. Since \( |f^{(4)}(x)| \leq 5x^2 \), it takes on its maximum at \( x = 2 \), so we may use \( K_4 = 5 \cdot 2^2 = 20 \). This means the maximum error in the above approximation is

\[
\frac{2^5 \cdot 20}{180(4)^4} = \frac{1}{72}.
\]

8. Does the improper integral \( \int_2^{\infty} \frac{x^{1/2} + 1}{5x - 6} \, dx \) converge? Fully justify your answer (note that if the integral converges, you need not give its value).

**Solution:** No, it diverges. Note that

\[
\frac{x^{1/2} + 1}{5x^2 - 6} > \frac{x^{1/2} + 1}{5x^2} > x^{1/2} > \frac{1}{5x^{1/2}}.
\]

But \( \frac{1}{5} \int_2^{\infty} \frac{dx}{x^{1/2}} \) diverges, since

\[
\int_2^{\infty} \frac{dx}{x^{1/2}} = \lim_{M \to \infty} 2x^{1/2} \bigg|_2^M = \lim_{M \to \infty} 2\sqrt{M} + 2\sqrt{2}
\]

which diverges to \( +\infty \) as \( M \to \infty \). By the comparison test, the original integral diverges.

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\(^1\)Feel free to use the fact that \( E_s < \frac{(b - a)^5}{180n^4} K_4 \), where \( K_4 = \max |f^{(4)}(x)| \) for \( a \leq x \leq b \).
Alternatively, many people did the following instead:

\[
\int_2^\infty \frac{x^{1/2} + 1}{5x - 6} \, dx = \int_2^\infty \frac{x^{1/2}}{5x - 6} \, dx + \int_2^\infty \frac{1}{5x - 6} \, dx \\
= \int_2^\infty \frac{x^{1/2}}{5x - 6} \, dx + \lim_{M \to \infty} \frac{1}{5} \ln \left| \frac{5M - 6}{5} \right| - \frac{\ln(4)}{5}
\]

Since the integral in the last term is positive and the limit diverges to \(+\infty\), the original integral must diverge. This method is, in fact, equivalent to comparison with \(\int_2^\infty \frac{dx}{5x - 4}\).

9. Find the area lying between the two curves \(x + y^2 = 0\) and \(x + y = 0\)

**Solution:** The curves in question are graphed at right. They intersect at \((0, 0)\) and at \((-1, 1)\): since the second curve is \(y = -x\), substituting this into the first curve gives \(x + x^2 = 0\), which holds when \(x = 0\) or \(x = -1\). We can choose to integrate either with respect to \(x\) or with respect to \(y\).

Integrating with respect to \(x\), we rewrite the curves as

\[
y = \pm \sqrt{-x} \quad y = -x.
\]

The upper curve is \(y = \sqrt{-x}\) and the lower curve is \(y = -x\), and the relevant \(x\)-values are \(-1 \leq x \leq 0\). The resulting integral is

\[
\int_{-1}^0 \sqrt{-x} + x \, dx = \frac{2}{3} (x)^{3/2} \bigg|_{-1}^{0} = \frac{2}{3} + \frac{1}{2} = \frac{1}{6}.
\]

If we choose instead to integrate with respect to \(y\), the rightmost curve is \(x = -y^2\) and the leftmost curve is \(x = -y\), and the relevant \(y\)-values are \(0 \leq y \leq 1\). So the integral is

\[
\int_0^1 -y^2 + y \, dy = \frac{-y^3}{3} \bigg|_0^1 + \frac{y^2}{2} \bigg|_0^1 = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6}.
\]