MAT 132 Solutions to Midterm 1 (acoustic)

$\boxed{\text{20 pts}}$ 1. $\displaystyle\int \frac{2x-3}{x^3-x}\,dx$

Solution: We do this by partial fractions. Since $x^3 - x = x(x-1)(x+1)$, we want to find numbers A, B, and C so that

$$\frac{2x-3}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}.$$

Cross-multiplying, we have $A(x-1)(x+1) + Bx(x+1) + Cx(x-1) = 2x - 3$.

$$\begin{aligned}
\text{If } x = 0, \text{ we have} \quad & -A = -3 && \text{so } A = 3 \\
\text{If } x = 1, \text{ we have} \quad & 2B = -1 && \text{so } B = -1/2 \\
\text{If } x = -1, \text{ we have} \quad & 2C = -5 && \text{so } C = -5/2.
\end{aligned}$$

This means that

$$\int \frac{2x-3}{x^3-x}\,dx = \int \frac{3}{x} - \frac{1/2}{x-1} - \frac{5/2}{x+1}\,dx = 3\ln|x| - \frac{1}{2}\ln|x-1| - \frac{5}{2}\ln|x+1| + C.$$

$\boxed{\text{20 pts}}$ 2. $\displaystyle\int_1^6 \frac{dr}{(r-2)^4}$

Solution: Since $\dfrac{1}{(r-2)^4}$ becomes undefined when $r = 2$, we have write the integral as the sum of two improper integrals. That is,

$$\begin{aligned}
\int_1^6 \frac{dr}{(r-2)^4} &= \int_1^2 \frac{dr}{(r-2)^4} + \int_2^6 \frac{dr}{(r-2)^4} \\
&= \lim_{t\to 2^-} \int_1^t \frac{dr}{(r-2)^4} + \lim_{t\to 2^+} \int_t^6 \frac{dr}{(r-2)^4} \\
&= \lim_{t\to 2^-} \left.\frac{-1}{3(r-2)^3}\right|_1^t + \lim_{t\to 2^+} \left.\frac{-1}{3(r-2)^3}\right|_t^6 \\
&= \lim_{t\to 2^-} \frac{-1}{3(t-2)^3} - \frac{1}{3} - \frac{1}{48} + \lim_{t\to 2^+} \frac{1}{3(t-2)^3}.
\end{aligned}$$

However, both of the limits above are undefined, so the integral diverges.

$\boxed{\text{20 pts}}$ 3. $\displaystyle\int_1^2 \frac{\sqrt{t^2-1}}{t}\,dt$

Solution: We make the substitution $t = \sec\theta$, and so $dt = \sec\theta\tan\theta\,d\theta$. Also, when $t = 1$ we have $\theta = 0$, and when $t = 2$ we have $\theta = \pi/3$. Thus, we have

$$\begin{aligned}
\int_1^2 \frac{\sqrt{t^2-1}}{t}\,dt &= \int_0^{\pi/3} \frac{\sqrt{\sec^2\theta-1}}{\sec\theta}\sec\theta\tan\theta\,d\theta = \int_0^{\pi/3}\sqrt{\tan^2\theta}\tan\theta\,d\theta \\
&= \int_0^{\pi/3}\tan^2\theta\,d\theta = \int_0^{\pi/3}(\sec^2\theta-1)\,d\theta \\
&= \quad \tan\theta-\theta\Big|_0^{\pi/3} = \tan(\pi/3)-\frac{\pi}{3} = \sqrt{3}-\frac{\pi}{3}.
\end{aligned}$$

$\boxed{\text{20 pts}}$ 4. $\displaystyle\int x^5\cos(x^3)\,dx$

Solution: First, we make the substitution $w = x^3$, so $dw = 3x^2\,dx$.

$$\begin{aligned}
\int x^5\cos(x^3)\,dx &= \frac{1}{3}\int w\cos w\,dw = \frac{1}{3}\left(w\sin w - \int \sin w\,dw\right) \\
&= \frac{1}{3}(w\sin w+\cos w)+C = \frac{x^3\sin x^3+\cos x^3}{3}+C
\end{aligned}$$

where above we used integration by parts, with $u = w$ and $dv = \cos w\,dw$, so $du = dw$ and $v = \sin w$.

$\boxed{\text{20 pts}}$ 5. $\displaystyle\int \sin^3(x)\cos^2(x)\,dx$

Solution: We use the identity $\cos^2 x+\sin^2 x = 1$ to transform all but one of the $\sin x$ terms into powers of $\cos x$. So we have

$$\int \sin^3(x)\cos^2(x)\,dx = \int (1-\cos^2 x)\cos^2 x\sin x\,dx.$$

Now we make the subsitution $u = \cos x$ with $du = -\sin x\,dx$ to get

$$\int (1-u^2)u^2\,du = \int (u^2-u^4)\,du = \frac{\cos^3 x}{3}-\frac{\cos^5 x}{5}+C.$$

20 pts

6. $\displaystyle\int_0^1 xe^{4x}\,dx$

Solution: We integrate by parts, with $u = x$ and $dv = e^{4x}\,dx$, so $du = dx$ and $v = \frac{1}{4}e^{4x}$. This gives us

$$\int_0^1 xe^{4x}\,dx = \left.\frac{x}{4}e^{4x}\right|_0^1 - \frac{1}{4}\int_0^1 e^{4x}\,dx = \left.\frac{x}{4}e^{4x} - \frac{1}{16}e^{4x}\right|_0^1 = \frac{e^4}{4} - \frac{e^4}{16} + \frac{1}{16}$$

7. The values of a function $f(x)$ are given by the table at right.

x	-0.5	0	0.5	1	1.5	2	2.5
$f(x)$	-1	0	3	2	1	-1	1

15 pts

(a) Use Simpson's rule to approximate $\displaystyle\int_0^2 f(x)\,dx$.

Solution: First, observe that there are two "extra" values in the table, namely $x = -0.5$ and $x = 2.5$. If we were to include those values (as some people did), the approximation would be for $\int_{-0.5}^{2.5} f(x)\,dx$.
We have $\Delta x = 1/2$, and $n = 4$. Simpson's rule is given by

$$S_4 = \frac{1/2}{3}\left(f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + f(2)\right) = \frac{1}{6}\left(0 + 12 + 4 + 4 - 1\right) = \frac{19}{6}.$$

5 pts

(b) If you know that $-5x^2 \le f^{(4)}(x) \le 5x^2$, what is the maximum error in the approximation above?[1]

Solution: This is just a matter of filling in the proper values into the given formula. Since $|f^{(4)}(x)| \le 5x^2$, it takes on its maximum at $x = 2$, so we may use $K_4 = 5 \cdot 2^2 = 20$. This means the maximum error in the above approximation is

$$\frac{2^5 \cdot 20}{180(4)^4} = \frac{1}{72}.$$

20 pts

8. Does the improper integral $\displaystyle\int_2^\infty \frac{x^{1/2}+1}{5x-8}\,dx$ converge? Fully justify your answer (note that if the integral converges, you need not give its value).

Solution: No, $\boxed{\textbf{it diverges.}}$ Note that $\dfrac{x^{1/2}+1}{5x^2-8} > \dfrac{x^{1/2}+1}{5x^2} > \dfrac{x^{1/2}}{5x^2} > \dfrac{1}{5x^{1/2}}$.
But $\displaystyle\frac{1}{5}\int_2^\infty \frac{dx}{x^{1/2}}$ diverges, since

$$\int_2^\infty \frac{dx}{x^{1/2}} = \lim_{M\to\infty} \left.2x^{1/2}\right|_2^M = \lim_{M\to\infty} 2\sqrt{M} + 2\sqrt{2}$$

which diverges to $+\infty$ as $M \to \infty$. By the comparison test, the original integral diverges.

[1] Feel free to use the fact that $E_s < \dfrac{(b-a)^5}{180n^4}K_4$, where $K_4 = \max|f^{(4)}(x)|$ for $a \le x \le b$.

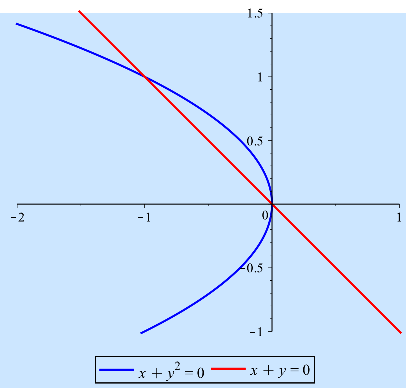
Alternatively, many people did the following instead:

$$\int_2^\infty \frac{x^{1/2}+1}{5x-8}\,dx = \int_2^\infty \frac{x^{1/2}}{5x-8}\,dx + \int_2^\infty \frac{1}{5x-8}\,dx$$
$$= \int_2^\infty \frac{x^{1/2}}{5x-8}\,dx + \lim_{M\to\infty} \frac{1}{5}\ln|5M-8| - \frac{\ln(2)}{5}$$

Since the integral in the last term is positive and the limit diverges to $+\infty$, the original integral must diverge. This method is, in fact, equivalent to comparison with $\int_2^\infty \frac{dx}{5x-4}$.

$\boxed{\text{20 pts}}$ 9. Find the area lying between the two curves $x + y^2 = 0$ and $x + y = 0$

Solution: The curves in question are graphed at right. They intersect at $(0,0)$ and at $(-1,1)$: since the second curve is $y = -x$, substituting this into the first curve gives $x + x^2 = 0$, which holds when $x = 0$ or $x = -1$. We can choose to integrate either with respect to x or with respect to y.



Integrating with respect to x, we rewrite the curves as

$$y = \pm\sqrt{-x} \qquad y = -x.$$

The upper curve is $y = \sqrt{-x}$ and the lower curve is $y = -x$, and the relevant x-values are $-1 \le x \le 0$. The resulting integral is

$$\int_{-1}^0 \sqrt{-x} + x\,dx = -\frac{2}{3}(-x)^{3/2} + \frac{x^2}{2}\Big|_{-1}^0 = -\frac{2}{3} + \frac{1}{2} = \frac{1}{6}.$$

If we choose instead to integrate with respect to y, the rightmost curve is $x = -y^2$ and the leftmost curve is $x = -y$, and the relevant y-values are $0 \le y \le 1$. So the integral is

$$\int_0^1 -y^2 + y\,dy = -\frac{y^3}{3} + \frac{y^2}{2}\Big|_0^1 = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6}.$$