Notes on Second Order Linear Differential Equations

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1. The general second order homogeneous linear differential equation with constant coefficients looks like

\[ Ay'' + By' + Cy = 0, \]

where \( y \) is an unknown function of the variable \( x \), and \( A, B, \) and \( C \) are constants. If \( A = 0 \) this becomes a first order linear equation, which we already know how to solve. So we will consider the case \( A \neq 0 \). We can divide through by \( A \) and obtain the equivalent equation

\[ y'' + by' + cy = 0 \]

where \( b = B/A \) and \( c = C/A \).

“Linear with constant coefficients” means that each term in the equation is a constant times \( y \) or a derivative of \( y \). “Homogeneous” excludes equations like \( y'' + by' + cy = f(x) \) which can be solved, in certain important cases, by an extension of the methods we will study here.

2. In order to solve this equation, we guess that there is a solution of the form

\[ y = e^{\lambda x}, \]

where \( \lambda \) is an unknown constant. Why? Because it works!

We substitute \( y = e^{\lambda x} \) in our equation. This gives

\[ \lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + c e^{\lambda x} = 0. \]

Since \( e^{\lambda x} \) is never zero, we can divide through and get the equation

\[ \lambda^2 + b\lambda + c = 0. \]

Whenever \( \lambda \) is a solution of this equation, \( y = e^{\lambda x} \) will automatically be a solution of our original differential equation, and if \( \lambda \) is not a solution, then \( y = e^{\lambda x} \) cannot solve the differential equation. So the substitution \( y = e^{\lambda x} \) transforms the differential equation into an algebraic equation!

Example 1. Consider the differential equation

\[ y'' - y = 0. \]

Plugging in \( y = e^{\lambda x} \) give us the associated equation

\[ \lambda^2 - 1 = 0, \]
which factors as
\[(\lambda + 1)(\lambda - 1) = 0;\]
this equation has \(\lambda = 1\) and \(\lambda = -1\) as solutions. Both \(y = e^x\) and \(y = e^{-x}\) are solutions to the differential equation \(y'' - y = 0\). (You should check this for yourself!)

**Example 2.** For the differential equation
\[y'' + y' - 2y = 0,\]
we look for the roots of the associated algebraic equation
\[\lambda^2 + \lambda - 2 = 0.\]
Since this factors as \((\lambda - 1)(\lambda + 2) = 0\), we get both \(y = e^x\) and \(y = e^{-2x}\) as solutions to the differential equation. Again, you should check that these are solutions.

3. For the general equation of the form
\[y'' + by' + cy = 0,\]
we need to find the roots of \(\lambda^2 + b\lambda + c = 0\), which we can do using the quadratic formula to get
\[\lambda = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.\]
If the discriminant \(b^2 - 4c\) is positive, then there are two solutions, one for the plus sign and one for the minus.

This is what we saw in the two examples above.

Now here is a useful fact about linear differential equations: if \(y_1\) and \(y_2\) are solutions of the homogeneous differential equation \(y'' + by' + cy = 0\), then so is the linear combination \(py_1 + qy_2\) for any numbers \(p\) and \(q\). This fact is easy to check (just plug \(py_1 + qy_2\) into the equation and regroup terms; note that the coefficients \(b\) and \(c\) do not need to be constant for this to work. This means that for the differential equation in Example 1 \((y'' - y = 0)\), any function of the form
\[pe^x + qe^{-x}\]
where \(p\) and \(q\) are any constants
is a solution. Indeed, while we can’t justify it here, all solutions are of this form. Similarly, in Example 2, the general solution of
\[y'' + y' - 2y = 0\]
is
\[y = pe^x + qe^{-2x},\]
where \(p\) and \(q\) are constants.

4. If the discriminant \(b^2 - 4c\) is negative, then the equation \(\lambda^2 + b\lambda + c = 0\) has no solutions, unless we enlarge the number field to include \(i = \sqrt{-1}\), i.e. unless we work with complex numbers. If \(b^2 - 4c < 0\), then since we can write any positive number as a square \(k^2\), we let \(k^2 = -(b^2 - 4c)\).
Then $ik$ will be a square root of $b^2 - 4c$, since $(ik)^2 = i^2k^2 = (-1)k^2 = -k^2 = b^2 - 4c$. The solutions of the associated algebraic equation are then

$$
\lambda_1 = \frac{-b + ik}{2}, \quad \lambda_2 = \frac{-b - ik}{2}.
$$

**Example 3.** If we start with the differential equation $y'' + y = 0$ (so $b = 0$ and $c = 1$) the discriminant is $b^2 - 4c = -4$, so $2i$ is a square root of the discriminant and the solutions of the associated algebraic equation are $\lambda_1 = i$ and $\lambda_2 = -i$.

**Example 4.** If the differential equation is $y'' + 2y' + 2y = 0$ (so $b = 2$ and $c = 2$ and $b^2 - 4c = 4 - 8 = -4$). In this case the solutions of the associated algebraic equation are $\lambda = (-2 \pm 2i)/2$, i.e. $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$.

**5.** Going from the solutions of the associated algebraic equation to the solutions of the differential equation involves interpreting $e^{\lambda x}$ as a function of $x$ when $\lambda$ is a complex number. Suppose $\lambda$ has real part $a$ and imaginary part $ib$, so that $\lambda = a + ib$ with $a$ and $b$ real numbers. Then

$$
e^{\lambda x} = e^{(a + ib)x} = e^{ax} e^{ibx}
$$

assuming for the moment that complex numbers can be exponentiated so as to satisfy the law of exponents. The factor $e^{ax}$ does not cause a problem, but what is $e^{ibx}$? Everything will work out if we take

$$e^{ibx} = \cos(bx) + i\sin(bx),$$

and we will see later that this formula is a necessary consequence of the elementary properties of the exponential, sine and cosine functions.

**6.** Let us try this formula with our examples.

**Example 3.** For $y'' + y = 0$ we found $\lambda_1 = i$ and $\lambda_2 = -i$, so the solutions are $y_1 = e^{ix}$ and $y_2 = e^{-ix}$. The formula gives us $y_1 = \cos x + i\sin x$ and $y_2 = \cos x - i\sin x$.

Our earlier observation that if $y_1$ and $y_2$ are solutions of the linear differential equation, then so is the combination $py_1 + qy_2$ for any numbers $p$ and $q$ holds even if $p$ and $q$ are complex constants.

Using this fact with the solutions from our example, we notice that $\frac{1}{2}(y_1 + y_2) = \cos x$ and $\frac{1}{2i}(y_1 - y_2) = \sin x$ are both solutions. When we are given a problem with real coefficients it is customary, and always possible, to exhibit real solutions. Using the fact about linear combinations again, we can say that $y = p\cos x + q\sin x$ is a solution for any $p$ and $q$. This is the general solution. (It is also correct to call $y = pe^{ix} + qe^{-ix}$ the general solution; which one you use depends on the context.)

**Example 4.** $y'' + 2y' + 2y = 0$. We found $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$. Using the formula we have

$$
y_1 = e^{\lambda_1 x} = e^{(-1+i)x} = e^{-x}e^{ix} = e^{-x}(\cos x + i\sin x),
$$

$$
y_2 = e^{\lambda_2 x} = e^{(-1-i)x} = e^{-x}e^{-ix} = e^{-x}(\cos x - i\sin x).
$$
Exactly as before we can take \( \frac{1}{2}(y_1 + y_2) \) and \( \frac{1}{2}(y_1 - y_2) \) to get the real solutions \( e^{-x}\cos x \) and \( e^{-x}\sin x \). (Check that these functions both satisfy the differential equation!) The general solution will be \( y = p e^{-x}\cos x + q e^{-x}\sin x \).

7. Repeated roots. Suppose the discriminant is zero: \( b^2 - 4c = 0 \). Then the “characteristic equation” \( \lambda^2 + b\lambda + c = 0 \) has one root. In this case both \( e^{\lambda x} \) and \( xe^{\lambda x} \) are solutions of the differential equation.

**Example 5.** Consider the equation \( y'' + 4y' + 4y = 0 \). Here \( b = c = 4 \). The discriminant is \( b^2 - 4c = 4^2 - 4 \times 4 = 0 \). The only root is \( \lambda = -2 \). Check that both \( e^{-2x} \) and \( xe^{-2x} \) are solutions. The general solution is then \( y = pe^{-2x} + qxe^{-2x} \).

8. Initial Conditions. For a first-order differential equation the undetermined constant can be adjusted to make the solution satisfy the initial condition \( y(0) = y_0 \); in the same way the \( p \) and the \( q \) in the general solution of a second order differential equation can be adjusted to satisfy initial conditions. Now there are two: we can specify both the value and the first derivative of the solution for some “initial” value of \( x \).

**Example 5.** Suppose that for the differential equation of Example 2, \( y'' + y' - 2y = 0 \), we want a solution with \( y(0) = 1 \) and \( y'(0) = -1 \). The general solution is \( y = pe^x + qe^{-2x} \), since the two roots of the characteristic equation are 1 and \(-2\). The method is to write down what the initial conditions mean in terms of the general solution, and then to solve for \( p \) and \( q \). In this case we have

\[
1 = y(0) = pe^0 + qe^{-2\times 0} = p + q
\]

\[-1 = y'(0) = pe^0 - 2qe^{-2\times 0} = p - 2q.
\]

This leads to the set of linear equations \( p + q = 1, p - 2q = -1 \) with solution \( q = 2/3, p = 1/3 \). You should check that the solution

\[
y = \frac{1}{3}e^x + \frac{2}{3}e^{-2x}
\]

satisfies the initial conditions.

**Example 6.** For the differential equation of Example 4, \( y'' + 2y' + 2y = 0 \), we found the general solution \( y = pe^{-x}\cos x + qe^{-x}\sin x \). To find a solution satisfying the initial conditions \( y(0) = -2 \) and \( y'(0) = 1 \) we proceed as in the last example:

\[
-2 = y(0) = pe^{-0}\cos 0 + qe^{-0}\sin 0 = p
\]

\[
1 = y'(0) = -pe^{-0}\cos 0 - pe^{-0}\sin 0 - qe^{-0}\sin 0 + qe^{-0}\cos 0 = -p + q.
\]

So \( p = -2 \) and \( q = -1 \). Again check that the solution

\[
y = -2e^{-x}\cos x - e^{-x}\sin x
\]

satisfies the initial conditions.