Test # 2

MAT 127 Spring 2005 solutions

1. A function $y(t)$ satisfies the differential equation

$$\frac{dy}{dt} = y - y^2.$$

(a) What are the equilibrium solutions?
Solution: The equilibrium solutions occur when $\frac{dy}{dt} = 0$, that is, when $y - y^2 = 0$. Factoring tells us $y = 0$ or $y = 1$.

(b) Rewrite the differential equation as a logistic equation. Find the growth rate k and the carrying capacity K.
Solution: Rewriting gives us

$$\frac{dy}{dt} = y(1 - y)$$

The growth rate $k = 1$, and the carrying capacity is also 1.

(c) If $y(0) = 3$ write the solution to the initial value problem.
Solution: I'll pretend we didn't memorize the formula which solves logistic equations. If you did, go ahead and use it. Since I forgot it, we separate variables to obtain

$$\int \frac{dy}{y(1-y)} = \int dt$$

Partial fractions gives us

$$\frac{1}{y(1-y)} = \frac{1}{1-y} - \frac{1}{y}$$

so we have

$$\int \frac{dy}{y(1-y)} = \int \frac{dy}{1-y} - \frac{dy}{y} = \ln(1-y) - \ln y = \ln \frac{1-y}{y}$$

and so we have

$$\ln \frac{1-y}{y} = t + C, \qquad \text{or, equivalently} \qquad \frac{1-y}{y} = Ae^t$$

Solving for y gives

$$y = \frac{1}{1 + Ae^{-t}}$$

Using the initial condition, we have

$$3 = \frac{1}{1 + A}$$

so $A = -2/3$, and the solution is

$$y = \frac{1}{1 - \frac{2}{3}e^{-t}}$$

2. Consider the sequence whose n^{th} term is $a_n = n2^{-n}$.

   (a) Show that after $n = 1$ the sequence is decreasing.

   Solution: One way to do this is to show that $f(x) = x2^{-x}$ is decreasing for $x > 1$. Since the derivative is $-2^{-x}(x \ln 2 - 1)$ and $x \ln 2 > 1$ for $x > 1/\ln 2$, we have $f'(x) < 0$ for $x \geq 1.442$. Hence the sequence is decreasing for $n \geq 2$.

   Alternatively, you could just show directly that $\frac{n}{2^n} > \frac{n+1}{2^{n+1}}$.

   (b) Assume the sequence converges and compute its limit.

   Solution: The limit is 0. We can use L'Hôpital's rule:

$$\lim_{n\to\infty} \frac{n}{2^n} = \lim_{n\to\infty} \frac{1}{2^n \ln 2} = 0$$

3. Compute the limits of the following convergent sequences:

   (a) $\left\{\frac{\ln(1+e^n)}{n}\right\}$

   Solution: Using L'Hôpital's rule:

$$\lim_{n\to\infty} \frac{\ln(1+e^n)}{n} = \lim_{n\to\infty} \frac{e^n/(1+e^n)}{1} = 1.$$

   (b) $\left\{\frac{(-5)^n}{n!}\right\}$

   Solution: Let's expand the form of a_n when n is large:

$$a_n = \frac{-5}{1} \cdot \frac{-5}{2} \cdot \frac{-5}{3} \cdot \frac{-5}{4} \cdot \frac{-5}{5} \cdot \frac{-5}{6} \cdots \frac{-5}{n}$$

   Notice that $|a_n|$ is the product of five numbers larger than one and $n - 5$ numbers less than $\frac{5}{6}$. So we know that

$$|a_n| \leq \frac{625}{24} \cdot \left(\frac{5}{6}\right)^{n-5} \qquad \text{for} \qquad n \geq 5$$

   But the limit of the right-hand side is certainly 0 as $n \to \infty$.

   The limit of the sequence is 0.

4. Consider the recursively defined sequence:

$$a_1 = 1 \qquad a_{n+1} = 6 - a_n \quad n \geq 1.$$

   (a) Does the sequence converge, diverge, or neither? Explain your answer.

   Solution: Let's look at a few terms and see what's going on. The sequence is $1, 5, 1, 5, \ldots$ Since $a_3 = a_1$ and a_n is defined interms of a_{n-1}, we know all the terms. This sequence oscillates forever between 1 and 5, and cannot converge.

   (b) What happens if the first term is $a_1 = 3$? Explain your answer.

   Solution: This time the sequence is $3, 3, 3, 3, \ldots$. Since every term is 3, the limit is 3.

5. Model a population of wild rabbits by assuming it obeys the logistic equation. When the population is very small it grows at a rate of $1/4$ its population size.

   (a) Initially there are 100 rabbits. After $4 \ln 4$ months there are 250 rabbits. Compute the carrying capacity of their population.

   Solution: We have

$$R'(t) = \frac{1}{4}R(t)\left(1 - \frac{R(t)}{L})\right)$$

   If we solve this as in the first problem, we get

$$R(t) = \frac{L}{1 + Ae^{-t/4}}$$

   Using the fact that $R(0) = 100$ and $R(4\ln 4) = 250$ tells us that

$$100 = \frac{L}{1+A} \qquad \text{and} \qquad 250 = \frac{L}{1 + A/4}$$

   Solving these gives $A = 4$, and so the carrying capacity L is 500.

   (b) When will there be 450 rabbits?

   Solution: We need to solve

$$450 = \frac{500}{1 + 4e^{-t/4}}$$

   for t. Doing the algebra gives

$$e^{-t/4} = \frac{1}{36}, \qquad \text{and taking logs gives} \qquad \frac{-t}{4} = -\ln 36$$

   This means it will take $4 \ln 36$ years, or about 14.3 years.