Final Exam
MAT 127 Spring 2005

Directions: There are 8 questions. You have until 1:30 PM (150 minutes). For credit, you must show all your work, using the backs of the pages if necessary. You may not use a calculator.

1. \frac{dy}{dt} = y^2 - y - 6.

(a) What are the constant solutions of the equation?
(b) For what values of \( y \) is \( y \) increasing?
(c) For what values of \( y \) is \( y \) decreasing?

A solution is constant if \( \frac{dy}{dt} = 0 \), so we solve
\[ \frac{dy}{dt} = 0 = y^2 - y - 6 = (y-3)(y+2) \]
Thus the constant solutions are \( y(t) = 3 \) and \( y(t) = -2 \).

\( y(t) \) is increasing when \( \frac{dy}{dt} > 0 \), which happens when \( y < -2 \) or \( y > 3 \).

\( \frac{dy}{dt} < 0 \) for \( -2 < y < 3 \), so \( y(t) \) decreases there.

The solutions look like this:

\[ y = 3 \]
\[ y = -2 \]
2. Solve the following initial value problems. (Hint: they are both separable.)

(a) \( \frac{dx}{dt} = -2x^2t, \ x(0) = 1/3 \)

(b) \( \frac{dy}{dt} = y \cos t, \ y(\pi) = 1 \)

\[ \text{A) } \frac{dx}{dt} = -2x^2t \]

\[ \int \frac{dx}{x^2} = \int 2t \, dt \]

\[ -\frac{1}{x} = -t^2 + C \]

\[ x(t) = \frac{1}{t^2 + C} \]

Since \( x(0) = 1/3 \), \( C = 3 \).

So \( x(t) = \frac{1}{t^2 + 3} \)

\[ \text{B) } \frac{dy}{\cos t} = y \, dt \]

\[ \int \frac{dy}{y} = \int \cos t \, dt \]

\[ \ln |y| = \sin t + C \]

Exponentiating, \( \sin t + C \)

So \( y = Ae^{\sin t} \) (where \( A = e^C \))

Since \( y(\pi) = 1 \),

\[ 1 = A e^{\sin \pi} = A \]

\[ y(t) = e^{\sin t} \]

3. Assume a contagious disease spreads at a rate proportional to the number of infected people. Initially there are 10 people infected and after 1 month there are 100 people infected.

(a) Find an expression for the number of infected people after \( t \) months.

(b) When will there be 1000 infected people?

Let \( P(t) \) be the number of people infected after \( t \) months.

Since the rate of increase of \( P(t) \) is proportional to \( P(t) \), we have

\[ P'(t) = kP(t) \]

\[ P(0) = 10, \ P(1) = 100 \]

So \( P(t) = Ae^{kt} \)

Since \( P(0) = 10 \), \( A = 10 \).

Since \( P(1) = 100 \), \( 10e^k = 100 \)

\[ 10 = e^k \]

So \( k = \ln 10 \)

\[ P(t) = 10e^{t \ln 10} \]

\[ (\text{or } P(t) = 10^{t+1}) \]

\[ \text{b) Since } P(t) = 10^{t+1} \]

1000 people will be infected when

\[ 1000 = 10^{t+1} \]

\[ 1000 = 10^{t+1} \]

That is, after

\[ t = \log_{10} 1000 - 1 \]

\[ t = 2 \] months.
4. Compute the limits of the following convergent sequences:

(a) \( \left\{ \frac{\sin 5n}{2n} \right\} \)

(b) \( \left\{ \frac{n}{(\ln n)^2} \right\} \)

\[ \lim_{n \to \infty} \frac{\sin 5n}{2n} \]

Since \(-1 \leq \sin 5n \leq 1\),

we have

\[ \frac{-1}{2n} \leq \frac{\sin 5n}{2n} \leq \frac{1}{2n} \]

So

\[ 0 \leq \lim_{n \to \infty} \frac{\sin 5n}{2n} \leq 0 \]

So the limit is 0 using the Squeeze Theorem.

\[ \lim_{n \to \infty} \frac{n}{(\ln n)^2} = \lim_{n \to \infty} \frac{1}{2 \ln n} \]

Using L'Hopital's Rule,

\[ = \lim_{n \to \infty} \frac{n}{2 \ln n} = \lim_{n \to \infty} \frac{1}{2 / n} \]

By L'Hopital's again,

\[ = \lim_{n \to \infty} \frac{n}{2} = \infty \]

So the sequence diverges to \( \infty \).
5. Determine whether the series is convergent or divergent. State which test you're using.

(a) $\sum_{n=1}^{\infty} \frac{3n}{n^3+4}$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+2}}$

We'll use limit comparison to compare

$\sum_{n=1}^{\infty} \frac{3n}{n^3+4}$ to $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$\lim_{n \to \infty} \frac{\frac{3n}{n^3+4}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{3n^3}{n^3+4} = 3$.

Since the limit is not 0 and it isn't $\infty$, the two series converge or diverge together.

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series, $p=2$),

so does $\sum_{n=1}^{\infty} \frac{3n}{n^3+4}$.

\[\square\]

\[b\] $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+2}}$ is an alternating series, so

we use the alt. series test.

$\lim_{n \to \infty} \frac{1}{\sqrt{n+2}} = 0$.

The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+2}}$ converges by alt. series test.
6. Compute the following sums.

(a) \[ \sum_{n=0}^{\infty} \frac{3^{2n+1}}{10^n} \]
(b) \[ \sum_{n=1}^{\infty} [\sin(1/n) - \sin(1/(n+1))] \]

\[ \sum_{n=0}^{\infty} \frac{3^{2n+1}}{10^n} = \sum_{n=0}^{\infty} \frac{3 \cdot 9^n}{10^n} \]

A geometric series with \( r = \frac{9}{10} \).

So the sum is \( \frac{3}{1-9/10} = 30 \).

Let's write out some terms:

\[ \sin(1) - \sin(\frac{1}{2}) + \sin(\frac{1}{3}) - \sin(\frac{1}{4}) + \ldots \]

Cancel terms:

\[ = \sin(1) \]

7. Compute and write out the following series. If applicable, you can use the table of Maclaurin series provided with your exam:

(a) The Maclaurin series for \( f(x) = x^3 e^{x^2} \).
(b) The Taylor series, centered around \( a = 1 \), for \( f(x) = x^3 \).

(a) \[ x^3 e^x = x^3 \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+3}}{n!} = x^3 + x^6 + x^9 + \ldots \]

(b) So the series is

\[ 1 + 3(x-1) + 3(x-1)^2 + (x-1)^3 \]
3. Consider the power series
\[ \sum_{n=1}^{\infty} \frac{(x-2)^n}{n^3}. \]

(a) Write the “center” of this power series.
(b) Find the open interval of absolute convergence.
(c) Determine whether the series converges or diverges at each of the interval’s endpoints.

\[ \textbf{a} \quad \text{THE INTERVAL OF CONVERGENCE IS ABOUT X = 2, SO THAT'S THE "CENTER".} \]

\[ \textbf{b} \quad \text{APPLYING THE RATIO TEST:} \]
\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(x-2)^{n+1}}{(n+1)^3} \cdot \frac{n^3}{(x-2)^n} \right| = \lim_{n \to \infty} \frac{n^3}{(n+1)^3} \cdot \left| x-2 \right| \]
\[ \left| x-2 \right| \lim_{n \to \infty} \frac{n}{n+1} = \frac{|x-2|}{3}. \]
\[ \text{THIS WILL BE < 1 WHEN} \quad \frac{|x-2|}{3} < 1, \]
\[ 1 \leq |x-2| < 3 \quad \text{AND} \quad -1 < x < 5. \]

\[ \textbf{c} \quad \text{NOW WE CHECK WHAT HAPPENS FOR} \quad x = 5 \quad \text{AN} \quad x = -1. \]
\[ \text{IF} \quad x = 5, \quad \text{THE SERIES IS} \quad \sum \frac{3^n}{n^3}, \quad \text{WHICH DIVERGES.} \]
\[ \text{IF} \quad x = -1, \quad \text{WE HAVE} \quad \sum \frac{(-3)^n}{n^3}, \quad \text{WHICH CONVERGES.} \]
\[ \text{SO THE INTERVAL OF CONV. IS} \quad [-1, 5). \]