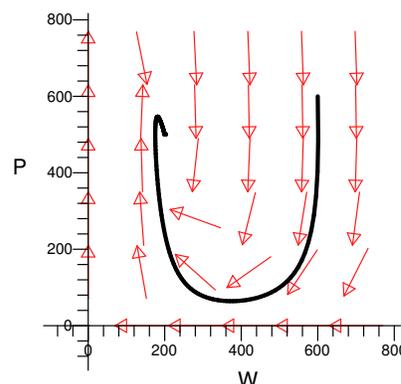


MAT 127 Solutions to Second Midterm

1. 20 points Two populations, the Pacifists and the Warriors, live near one another. The Pacifists are simple rutabaga farmers: if left to themselves, their population would be well modelled by a logistic growth model. However, the Warriors live nearby, and they survive by making periodic raids on the Pacifists. The two populations are modelled by the predator-prey system below, where t is in years, $W(t)$ is the population of the Warriors after t years, and $P(t)$ is the population of the Pacifists. The phase portrait for this system is shown at right.



$$\frac{dP}{dt} = 2P \left(1 - \frac{P}{1000} \right) - \frac{PW}{200}$$

$$\frac{dW}{dt} = -\frac{W}{4} + \frac{PW}{2000}$$

- (a) Are there any equilibrium solutions? If so, find **all** of them. If not, write “none”, and **justify your answer**.

Solution: We will have an equilibrium solution when both $\frac{dP}{dt}$ and $\frac{dW}{dt}$ are zero simultaneously. So, first we set them to zero and factor, getting:

$$0 = 2P \left(1 - \frac{P}{1000} - \frac{W}{400} \right)$$

$$0 = \frac{W}{4} \left(-1 + \frac{P}{500} \right)$$

The second equation will be zero if $W = 0$ or $P = 500$. If $W = 0$, then the first equation becomes

$$0 = 2P \left(1 - \frac{P}{1000} \right)$$

and so either $P = 0$ or $P = 1000$. This makes sense, since if there are no Warriors ($W = 0$), the population of Pacifists obeys a logistic model.

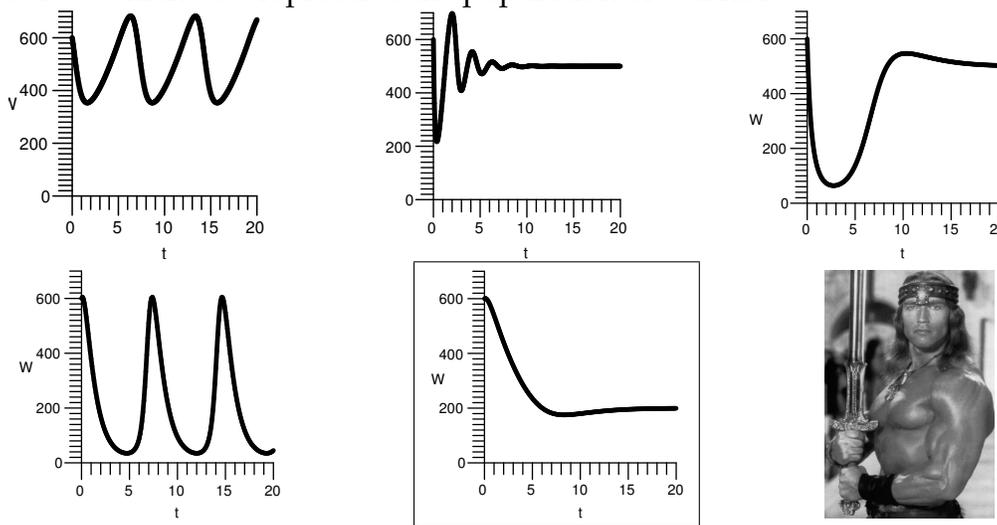
If $W \neq 0$, then to have an equilibrium, we must have $P = 500$. In this case, the first equation becomes

$$0 = 1000 \left(1 - \frac{500}{1000} - \frac{W}{400} \right) \quad \text{that is,} \quad 0 = 1000 \left(\frac{1}{2} - \frac{W}{400} \right)$$

and so we see we must have $W = 200$.

Thus, there are three equilibrium points: $(W = 0, P = 0)$, $(W = 0, P = 1000)$, and $(W = 200, P = 500)$.

- (b) If the populations start out with 600 Pacifists and 600 Warriors, circle the graph below which best represents the population of Warriors.



Solution: In order to see this, first look at the trajectory in the phase portrait. The phase portrait has W on the horizontal axis, and P vertical, and the indicated trajectory starts at $(600, 600)$, then goes through about $(400, 100)$ and then P starts to increase while W continues to drop. At about $(180, 550)$, there is a little hook, where the trajectory turns and heads towards the equilibrium at $(200, 500)$.

The graph we want is for W as a function of t . This means W needs to start at $W = 600$, then decrease to just under $W = 200$, and increase back towards $W = 200$. The middle graph on the second row is the only one that does this.

(The photo of Conan the Barbarian may best represent the look of a typical warrior, but it isn't a graph of a function. Imagine trying to get Conan to pass the "vertical line test"!)

2. 20 points For each of the **sequences** below, determine if it converges or diverges. If the sequence converges, state its limit.

(a) $\left\{ \frac{n^2 - 4}{n^2 + 2n} \right\}_{n=1}^{\infty}$

Solution: Here we just do a standard limit problem:

$$\lim_{n \rightarrow \infty} \frac{n^2 - 4}{n^2 + 2n} = \lim_{n \rightarrow \infty} \frac{1 - 4/n^2}{1 + 2/n} = \frac{1 - 0}{1 + 0} = 1$$

(b) $\{\arctan(n!)\}_{n=1}^{\infty}$

Solution: Another "regular" limit. We just have to remember how $\arctan x$ behaves for large x :

$$\lim_{n \rightarrow \infty} \arctan(n!) = \lim_{x \rightarrow \infty} \arctan(x) = \frac{\pi}{2}$$

(c) $\left\{ \frac{n \sin(n)}{2n-3} \right\}_{n=1}^{\infty}$

Solution: For n very large, $\frac{n}{2n-3}$ is very close to $\frac{1}{2}$, so this sequence will behave like $\frac{1}{2} \sin(n)$. However, the sine oscillates between -1 and 1 , so there can be no limit.

(d) The sequence $\{a_n\}_{n=1}^{\infty}$ with $a_0 = 0$ and $a_{n+1} = (a_n)^2 - 2$.

Solution: The best thing to do here is write out some terms and get a feel for what is going on. We have

$$a_0 = 0, \quad a_1 = 0^2 - 2 = -2, \quad a_2 = (-2)^2 - 2 = 4 - 2 = 2, \quad a_3 = 2^2 - 2 = 2.$$

Since each term is calculated from the previous one, we can see that all terms after a_3 will also be 2, so the limit of the sequence is 2.

3. 20 points Find the sums of the convergent series below:

(a) $\sum_{n=0}^{\infty} (-1)^n \frac{2^n}{3^n} = 1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \dots$

Solution: This is a geometric series with ratio $-\frac{2}{3}$. So we have

$$\sum_{n=0}^{\infty} (-1)^n \frac{2^n}{3^n} = \sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^n = \frac{1}{1 - \left(-\frac{2}{3}\right)} = \frac{3}{5}$$

(b) $\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \frac{2}{3} + \frac{1}{4} + \frac{2}{15} + \dots$ (Hint: partial fractions might be helpful.)

Solution: Using the hint, we look for A and B so that $\frac{2}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$.

That means we must have

$$2 = A(n+2) + Bn.$$

Setting $n = 0$ and $n = -2$, we have

$$2 = 2A \quad \text{so} \quad A = 1 \quad \text{and} \quad 2 = -2B \quad \text{so} \quad B = -1$$

Thus,

$$\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2} \right) = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \dots$$

Everything except $1 + \frac{1}{2}$ cancels out, so we have

$$\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = 1 + \frac{1}{2} = \frac{3}{2}$$

4. 20 points For each of the series below, decide if it converges or diverges. You do not need to find the value of the sum. **You must justify your answer to receive full credit.**

(a)
$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 2} = \frac{1}{3} + \frac{1}{6} + \frac{1}{11} + \frac{1}{18} + \dots$$

Solution: Since $\frac{1}{n^2 + 2} < \frac{1}{n^2}$, we have $\sum_{n=1}^{\infty} \frac{1}{n^2 + 2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$. But the latter is a p -series with $p = 2$, and so it converges. Hence the original series also converges using the comparison test.

(b)
$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} = \frac{1}{2 \ln 2} + \frac{1}{3 \ln 3} + \frac{1}{4 \ln 4} + \dots$$

Solution: Here, the integral test is appropriate. We need to determine if the corresponding improper integral is finite. So, we have

$$\int_2^{\infty} \frac{dx}{x \ln x} = \int_{\ln 2}^{\infty} \frac{du}{u} = \lim_{N \rightarrow \infty} (\ln(N) - \ln(\ln 2)) = \infty$$

Since the integral diverges, so does the series.

5. 20 points A population of rabbits is moved to an island with lots of grass and wild carrots and no predators. Assume the number of rabbits $R(t)$ in each year is well described by a logistic model

$$R'(t) = kR(t) \left(1 - \frac{R(t)}{L} \right)$$

where k and L are appropriate constants.

- (a) Suppose that the carrying capacity of the island is 1000 rabbits, and initially there were 250 rabbits. At the end of the first year, there were 400 rabbits. Find an expression for $R(t)$, the number of rabbits after t years.

Solution: I'll assume that you remembered the formula for the solution to a logistic equation, as most people did. The carrying capacity is L , and is given to be 1000. So, we have

$$R(t) = \frac{1000}{1 + Ae^{-kt}}$$

Since we initially had 250 rabbits, this means

$$R(0) = 250 = \frac{1000}{1 + A} \quad \text{so} \quad A = 3.$$

Since there were 400 rabbits at the end of the first year, we have

$$R(1) = 400 = \frac{1000}{1 + 3e^{-k}} \quad \text{so} \quad 1 + 3e^{-k} = \frac{5}{2} \quad \text{and} \quad e^{-k} = \frac{1}{2}$$

Taking logs gives us $k = \ln 2$ (you might have $k = -\ln \frac{1}{2}$, which is equivalent). Putting it all together, we have

$$R(t) = \frac{1000}{1 + 3e^{-t \ln 2}}$$

(b) How many years will it take the population to reach 750 rabbits?

Solution: Using our solution from the first part, we must find t so that

$$750 = \frac{1000}{1 + 3e^{-t \ln 2}}$$

So,

$$1 + 3e^{-t \ln 2} = \frac{4}{3} \quad \text{or} \quad e^{-t \ln 2} = \frac{1}{9}$$

Taking the log of both sides gives

$$-t \ln 2 = \ln \frac{1}{9} \quad \text{or} \quad t \ln 2 = \ln 9 \quad \text{i.e.} \quad t = \frac{\ln 9}{\ln 2}$$

So, it will take $\ln 9 / \ln 2$ years, or just over 3 years and 2 months to have 750 rabbits.